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1995 J. Phys. A: Math. Gen. 28 1069

(http://iopscience.iop.org/0305-4470/28/4/028)

J. Phys. A: Math. Gen. 28 (1995) 1069–1088. Printed in the UK

On the two-gap locus for the elliptic Calogero–Moser model

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Received 30 March 1994, in final form 7 December 1994

Abstract. We give an analytical description of the locus of the two-gap elliptic potentials associated with the corresponding flow of the Calogero–Moser system. We start with the description of Treibich–Verdier two-gap elliptic potentials. The explicit formulae for the covers, wavefunctions and Lamé polynomials are derived, together with a new Lax representation for the particle dynamics on the locus. We then consider more general potentials within the Weierstrass reduction theory of theta functions to lower genera. The reduction conditions in the moduli space of the genus-2 algebraic curves are given. This is a subvariety of the Humbert surface, which can be singled out by the condition of the vanishing of some theta constants.

1. Introduction

The Calogero–Moser model, whose complete integrability was shown a number of years ago (cf [22]), continues to attract more and more attention. This model has a rich algebraic-geometrical structure: its flows are connected with the pole dynamics of elliptic solutions of completely integrable partial differential equations [1], the Lax representation for the model depends through elliptic functions on the spectral parameter [20], and only the integration in terms of zeros of theta functions for the model is known [20]. The system permits a relativistic generalization, which is also completely integrable [23].

The classical Poisson r-matrix structure for the elliptic Calogero–Moser model was described very recently [7, 24]. The r-matrix found appears to be of linear dynamical type, i.e. dependent on the dynamical variables. The classical Poisson structure for the relativistic galvanization of the Calogero–Moser model is described only in the soliton case with a quadratic r-matrix of dynamical type [2]. The separated variables for these systems remain an unsolved problem besides the case of a small number of particles (see, for example, [8]).

The quantum Calogero–Moser problem also has a rich algebraic structure [13]. It is remarkable that the solutions of the quantum problem are isomorphic to the solutions of the Knizhnik–Zamolodchikov equations which are now understood to play an important role in the theory of quantum integrable models [26].

Because the Calogero–Moser model describes the pole dynamics for the elliptic solutions of the Kadomtsev–Petviashvili-type equations [1], its elliptic case becomes the classically known Lamé potentials of the Schrödinger equation. Although this paper is devoted to the investigation of elliptic potentials of the one-dimensional Schrödinger equation, we emphasize the importance of such potentials for different problems: the finite-gap multidimensional spectral problem [30], the Wess–Zumino–Witten model on the torus [12] and others.

0305-4470/95/041069+20$19.50

All the results given below can be generalized to higher genera, but we shall restrict ourselves to the investigation of the first non-trivial case of genus-2 to give the more complete formulae. Throughout the paper we have used computer algebra systems (Mathematica [31] and Maple [11]) to derive and simplify the formulae.

The paper is organized as follows. In section 2 we discuss the linear r-matrix algebra for the Calogero–Moser system and define its restriction to the locus associated with the KdV dynamics. In section 3 we describe the two-gap Lamé and Treibich–Verdier potentials [27, 28] for which we find explicitly the covers over the tori, derive the wavefunctions of the associated Schrödinger equations and Lamé polynomials. We also give a new Lax representation for the dynamics of particles on the locus in terms of 2×2 matrices. We show in section 4 that Treibich–Verdier potentials are special cases of elliptic potentials. Using the classical reduction theory of Riemann theta functions to lower genera (see, for example, [4, 5, 18]), we give necessary and sufficient conditions under which the two-gap potential is elliptic. We formulate these conditions in terms of vanishing of some theta constants which, in turn, are some subvarieties of Humbert surfaces (see, for example, [18, 29]). We derive one of the two-gap Treibich–Verdier potentials from this theta functional approach and give a new example of an elliptic potential. The paper is supported by two appendices which contain the description of spectral characteristics of the Treibich–Verdier potentials and all the necessary formulae to complete the theta functional computations mentioned in the paper.

2. The Calogero–Moser system on the locus

The elliptic Calogero–Moser model is the system of N one-dimensional particles interacting via a two-particle potential described by the Hamiltonian

$$H=\sum_i y_i^2+\sum_{i,j}\wp(x_i-x_j) \tag{2.1}$$

with $\wp$ being the Weierstrass elliptic function [3] with the periods 2ω, $2\omega'$ and y_i, x_i $(u=1,\dots,N)$ being canonical variables, $\{y_i,y_j\}=\{x_i,x_j\}=0$, $\{y_i,x_j\}=\delta_{ij}$.

Let $\{X_\mu\}=\{H_i,E_\alpha\}$, be basis matrices, $H_i=(\delta_{ij}\delta_{ik})$, $i=1,\dots,N$, $E_\alpha=E_{nm}=(\delta_{nj}\delta_{nk})$, $n\neq m$, $m,n=1,\dots,N$.

The Lax operator of the system L was found by Krichever and has the form [20]

$$L(u)=\sum_j y_jH_j+\mathrm{i}\sum_\alpha \Phi_\alpha E_\alpha \tag{2.2}$$

where

$$\Phi_\alpha=\Phi(\boldsymbol{x}\cdot\boldsymbol{\alpha};u) \qquad \Phi(x;u)=\frac{\sigma(u-x)}{\sigma(x)\sigma(u)}\mathrm{e}^{\zeta(u)x} \tag{2.3}$$

where σ and ζ are Weierstrass functions. The Hamiltonian flows of the system are generated by $\operatorname{Tr}L^n$, in particular, $\operatorname{Tr}L^2$ gives the Hamiltonian (2.1).

The Poisson structure of the system, as recently shown by Sklyanin [24] and Braden and Suzuki [7], is described by a linear dynamical r-matrix algebra,

$$\{L_1(u),L_2(v)\}=[r_{12}(u,v),L_1(u)]-[r_{21}(u,v),L_2(v)] \tag{2.4}$$

where $L_1=L\otimes I$, $L_2=I\otimes L$, $r_{12}(u,v)=\sum_{\mu,\nu}r^{\mu\nu}(u,v)X_\mu\otimes X_\nu$ is an $N^2\times N^2$ matrix depending on the dynamical variables, and $r_{21}(u,v)=Pr_{12}(v,u)P$, P is the permutation:

$Px \otimes y = y \otimes x$. The non-zero elements of the r-matrix are [7]

$$\begin{aligned} &r^{-\alpha\alpha}(u,v) = \Phi_\alpha(v-u)\mathrm{e}^{\psi(u,v)} \qquad r^{i\alpha}(u,v) = \tfrac{1}{2}\Phi_\alpha(v) \\ &r^{ij}(u,v) = \psi(u,v)\delta_{ij} \\ &\psi(u,v) = \zeta(v-u) + \zeta(u) - \zeta(v)\,. \end{aligned} \tag{2.5}$$

The r-matrix satisfies the dynamical Yang–Baxter equation,

$$\begin{aligned} &[d_{12}(x,y), d_{13}(x,z)] + [d_{12}(x,y), d_{23}(y,z)] + [d_{32}(z,y), d_{13}(x,z)] + \{L_2(y) \overset{\otimes}{,} d_{13}(x,z)\} \\ &\qquad -\{L_3(z) \overset{\otimes}{,} d_{12}(x,y)\} + [S_{13}(x,z), L_2(y)] - [S_{12}(x,y), L_3(z)] = 0 \end{aligned} \tag{2.6}$$

where the two other equations are obtained by cyclic permutations and in this context $d_{12}(x,y) = r_{12}(x,y) \otimes I$, $d_{23}(y,z) = I \otimes r_{23}(y,z)$, $d_{13}(x,y) = \sum_{\mu,\nu} r^{\mu\nu}(x,y) X_\mu \otimes I \otimes X_\nu$, and the S-matrix has the form

$$\begin{aligned} S_{13}(x,z) &= -\Phi_\alpha(x-z)\exp\psi(x,z) E_{-\alpha} \otimes H_i \otimes E_\alpha \\ S_{12}(x,y) &= -\Phi_\alpha(x-y)\exp\psi(x,y) E_{-\alpha} \otimes E_\alpha \otimes H_i\,. \end{aligned} \tag{2.7}$$

We point out that the S-matrix (2.7) differs from that given by Sklyanin [24], where a different representation for the operator L was considered. Although the S-term already appeared in [9], its significance became more evident after [24].

The equation

$$\det(L - \lambda I) = 0 \tag{2.8}$$

defines the *Krichever curve* i.e. the algebraic curve $C_N = (\lambda, u)$ which is an N-sheeted cover of a torus in $\pi : C_N \to C_1$

$$\lambda^N + \sum_{i=0}^{N-1} r_i(u)\lambda^{N-i-1} = 0 \tag{2.9}$$

where $r_i(u)$ are elliptic functions. In particular, the first two of them are given by $r_1(u) = -\binom{N}{2}\wp(u) + \sum \wp_{ij}$, $r_2(u) = \binom{N}{3}\wp'(u)$.

We consider the restriction of the third flow, $\mathrm{Tr}\, L^3$ of the Calogero–Moser system to the variety of stable points of the second flow, grad H—the *locus* $\mathcal{L}_N$,

$$\mathcal{L}_N = \left\{ (\boldsymbol{x}, \boldsymbol{y}) \middle| y_i = 0, \quad \sum_{i \neq j} \wp'(x_i - x_j) = 0,\ x_i \neq x_j,\ i, j = 1, \ldots, N \right\}. \tag{2.10}$$

It is shown in [1] that if the particles x_i move over the locus according to the equation

$$\frac{\mathrm{d}x_i}{\mathrm{d}t} = -12 \sum_{j=1, j\neq i}^{N} \wp(x_i - x_j) \qquad i = 1, \ldots, N \tag{2.11}$$

then

$$u(x) = 2\sum_{j=1}^{N} \wp(x - x_j(t)) + C \tag{2.12}$$

is an elliptic solution of the KdV equation $u_t = 6uu_x - u_{xxx}$ where C is a constant.

The geometry of the locus $\mathcal{L}_N$ was studied by Airault *et al* [1] and others. They showed that the locus is non-empty for positive triangle integers N, i.e. for numbers of the form $N = g(g+1)/2$, where g is the number of gaps in the spectrum (or the genus of the corresponding algebraic curve). The corresponding elliptic potential is the g-gap Lamé potential. Recently Treibich and Verdier [28] found a new set of elliptic potentials of the form (2.12) corresponding to non-triangle numbers of points on the locus $\mathcal{L}_N$. In particular, for the points of the locus x_i being the half-periods they found a family of elliptic potentials of the form [27]

$$u(x)=\sum_{i=0}^{3} g_i(g_i+1)\wp(x-\omega_i) \qquad g_i\in\mathbb{N} \tag{2.13}$$

which are associated with the cover of degree $N=\frac{1}{2}\sum_{i=0}^{3} g_i(g_i+1)$ over a torus. We shall refer to these potentials as *Treibich–Verdier potentials*.

The curve (2.8) becomes hyperelliptic when restricted to the $\mathcal{L}_N$ [4]. Therefore one expects to be able to write down a 2×2 Lax representation for the particle dynamics on a locus. This is done below for the two-gap Lamé and Treibich–Verdier potentials. Nevertheless the r-matrix formulation of the Calegero-Moser flows restricted to the locus remains an unsolved problem.

3. Two-gap Treibich–Verdier potentials

3.1. The spectral characteristics of elliptic solitons

We shall start on the potential of the form (2.13). There exist exactly six two-gap Treibich–Verdier potentials $u_N(x)$ associated with N-sheeted covering of the torus (shown in table 1).

We note that the three last potentials are simply the two-order transformation (Gauss transformation) of the first two potentials

$$\wp\left(z|\omega,\tfrac{1}{2}\omega'\right)=\wp(z)+\wp(z+\omega'). \tag{3.1}$$

Therefore we shall refer to the first three potentials as *primitive*.

To describe the two gap Lamé potential $6\wp(x)$ and primitive Treibich–Verdier potentials we have to

- exhibit the associated algebraic curve of genus-2;

$$C_2=(w,z),\ w^2=\prod_{i=1}^{5}(z-z_i) \tag{3.2}$$

- give its covers $\pi: C_2\to C_1$ and $\widetilde{\pi}: C_2\to\widetilde{C}_1$ over the tori $C_1=(\wp',\wp)$, $(\wp')^2=4\wp^3-g_2\wp-g_3$ and $\widetilde{C}_1=(\widetilde{\wp}',\widetilde{\wp})$, $(\widetilde{\wp}')^2=4\widetilde{\wp}^3-\widetilde{g}_2\widetilde{\wp}-\widetilde{g}_3$, where the moduli $\widetilde{g}_2$, $\widetilde{g}_3$ are expressed in some way through the moduli g_2, g_3;
- describe the two-gap locus $\mathcal{L}_N$;
- write the solution Ψ of the Schrödinger equation.

We can do all this by classical means (which modern computer algebra makes more effective) following the work of Hermite [15] and Halphen [14].

Table 1. Six two-gap Treibich–Verdier potentials.

N	$u_N(x)$
3	$6\wp(x)$ or $2\wp(x+\omega_1)+2\wp(x+\omega_2)+2\wp(x+\omega_3)$
4	$6\wp(x)+2\wp(x+\omega_i)$, $i=1,2,3$
5	$6\wp(x)+2\wp(x+\omega_i)+2\wp(x+\omega_j)$, $i\neq j=1,2,3$
6	$6\wp(x)+6\wp(x+\omega_i)$, $i=1,2,3$
8	$6\wp(x)+6\wp(x+\omega_i)+2\wp(x+\omega_j)+2\wp(x+\omega_k)$, $i\neq j\neq k$
12	$6\wp(x)+6\wp(x+\omega_1)+6\wp(x+\omega_2)+6\wp(x+\omega_3)$

Let us consider the Lamé equation

$$\left[\frac{\partial^2}{\partial x^2} - \sum_{i=1}^{n} a_i(a_i+1)\wp(x-x_i)\right]\Psi(x;u) = z\Psi(x;u) \tag{3.3}$$

where $\sum_{i=1}^{n} \frac{1}{2}a_i(a_1+1) = N$ is the degree of the cover.

We shall use the following generalization of the Hermite [15] and Halphen [14] ansatz for the function Ψ:

$$\Psi(x;u) = \mathrm{e}^{\lambda x} \sum_{i=1}^{n} \sum_{j=0}^{a_i-1} A_j(z,\lambda,u) \frac{\partial^j}{\partial x^j}\Phi(x-x_i,u) \tag{3.4}$$

where the function $\Phi(x;u)$ is the solution of (3.3) for $n=1$, $a_1=1$ is given by (2.3) and $A_j(z,\lambda,u)$ are some functions of the spectral parameters z, λ and u. Although the ansatz is valid for any point of the locus $\mathcal{L}_N$ we shall consider below only special points of the form $x_i = \omega_i$ or 0 found in [27] and listed previously in table 1. We shall refer to the *Lamé* polynomials $\Lambda_k(x)$ as the values of $\Psi(x;u)$ at values of u corresponding to the edges of the gaps $u = u_k$, $k = 1, \ldots, 5$.

After substituting the expansions of $\Phi(x,u)$ near the pole at $x=0$,

$$\Phi(x,u) = \frac{1}{x} - \frac{\wp(u)}{2}x + \frac{\wp'(u)}{6}x^2 + \frac{g_2 - 5\wp^2(u)}{40}x^3 + \cdots \tag{3.5}$$

and near $x = \omega_i$

$$\Phi(x+\omega_i) = \Phi(\omega_i)\left(1 + \frac{\wp'(u)}{2(e_i - \wp(u))}x + \tfrac{1}{2}(2e_i + \wp(u))x^2 + \cdots\right) \tag{3.6}$$

to (3.3) and equating the principal parts of the poles we come to an overdetermined linear system for A_i. The compatibility conditions give exactly two conditions

$$\mathcal{P}_1(\lambda, z, \wp(u)) = 0 \qquad \mathcal{P}_2(\lambda, z, \wp(u)) = 0 \tag{3.7}$$

with polynomials $\mathcal{P}_i$ of their arguments. By eliminating the variables z or $\wp$ from the conditions (3.7) we obtain two equivalent realizations of the curve (3.2); eliminating the variable z we obtain the first cover.

To find the second cover we use the fact that there exists the reduction formula

$$\frac{\mathrm{d}z}{w} = -\frac{\mathrm{d}\widetilde{\wp}}{\widetilde{\wp}'} \tag{3.8}$$

with $(\widetilde{\wp}', \widetilde{\wp})$ lying on the torus $\widetilde{C}_1$ and the coordinate $\widetilde{\wp}$ being a rational function of z,

$$\wp = \frac{Q_N(z)}{P_{N-3}(z)} \tag{3.9}$$

where Q_N and P_{N-3} are polynomials of orders N and $N-3$, respectively.

The description of the spectral characteristics of the primitive Treibich–Verdier potentials is given in appendix A.

3.2. *The dynamics on the locus*

The complete description of the dynamics on the locus under the action of the KdV flow was given in [1] for the case of the two-gap Lamé potential by some tricky manipulations with (2.10) and (2.11). It was shown that the dynamics are described by a foliation where the basis and the bundle are, respectively, the elliptic curves C_1 and $\widetilde{C}_1$ whose moduli are inter-dependent. The paper also conjectured that the same foliation would occur for all two-gap elliptic potentials.

We show below how to compute the second curve $\tilde{C}_1$ for primitive Treibich–Verdier potentials. The statement of [1] about the foliation can be proved by means of the Weierstrass reduction theory [4, 18] in the next section.

To describe the dynamics over the locus we write the Jacobi inversion problem, for the curve associated with elliptic potential

$$\int_\infty^{\mu_1} \frac{z\,\mathrm{d}z}{w} + \int_\infty^{\mu_2} \frac{z\,\mathrm{d}z}{w} = 2\mathrm{i}x + C_1 \qquad \int_\infty^{\mu_1} \frac{\mathrm{d}z}{w} + \int_\infty^{\mu_2} \frac{\mathrm{d}z}{w} = -8\mathrm{i}t + C_2 . \tag{3.10}$$

From the *trace formulae* [32] written for the elliptic potential in the form

$$\mu_1 + \mu_2 = -\sum_{j=1}^{N} \wp(x - x_j) + \tfrac{1}{2}\sum_{j=1}^{5} z_j$$

$$\mu_1\mu_2 = 3\sum_{i<j} \wp(x - x_i)\wp(x - x_j) - \tfrac{1}{8}Ng_2 + \tfrac{1}{2}\sum_{i<j} z_i z_j - \tfrac{3}{8}\left(\sum_{j=1}^{5} z_j\right)^2$$

we find in the vicinity of the point x_j the decompositions

$$\mu_1(x_j + \varepsilon, t) = \frac{1}{\varepsilon^2} + \mathrm{o}(1) \qquad \mu_2(x_j + \varepsilon, t) = -3\left(\sum_{i\neq j} \wp(x_j - x_i)\right) + \mathrm{o}(\varepsilon) . \tag{3.11}$$

Therefore the equations (3.10) in which $x = x_j$ and integrals are hyperelliptic are expressed in terms of elliptic functions in the following way:

$$\mathcal{Q}_1(\mu_1(x_j)) = \wp(ax_j + bt + c) \qquad \mathcal{Q}_2(\mu_1(x_j)) = \tilde{\wp}(dt + e) \tag{3.12}$$

where $\mathcal{P}_{1,2}$ are rational functions of the Nth degree, $\wp$ and $\tilde{\wp}$ are Weierstrass elliptic functions defined on the first and second tori, respectively; a, b, c, d and e are constants that appear under reduction. By eliminating the variable μ_1 from (3.11), we have an algebraic equation of the Nth degree with respect to $\wp$ and coefficients depending on $\tilde{\wp}$.

In particular, we have the following isospectral deformation of the potentials u_3, u_4 and u_5. Let

$$X_j = -3\sum_{k\neq j}^{N} \wp(x_j - x_k) \qquad j = 1, \ldots, N .$$

Then we have for $N = 3, 4$ and 5, respectively,

$$u_3 : 4X^3 - 9g_2X + 9g_3 + \tfrac{16}{9}\tilde{\wp}(8\mathrm{i}t) = 0 \tag{3.13}$$

$$u_4 : 9(X - z_2)(X - z_3)(X + 4e_i - e_k)^2 + 4(X + 6e_i)(\tilde{\wp}(8\mathrm{i}t) - \tilde{e}_j) = 0 \tag{3.14}$$

$$u_5 : 9P_5(X) + 4(X - 3e_i - 9e_j)(X - 3e_i - 9e_k)\tilde{\wp}(8\mathrm{i}t) = 0 \tag{3.15}$$

where the polynomial $P_5(z)$ in (3.15) is given in table A3.

We note that the rational limit of the dynamics is the same for all the potentials. The equations (3.13)–(3.15) give the integration of the corresponding Calogero–Moser flows restricted to the locus.

We also note that (3.13) can be extracted from [1, p 144].

Let us construct the Lax representation for Calogero–Moser system, being restricted to the locus. We choose the ansatz for such a representation in the form of 2×2 matrices

$$\begin{aligned} \dot{L}(z) &= [M(z), L(z)] \\ L(z) &= \begin{pmatrix} V(z) & U(z) \\ W(z) & -V(z) \end{pmatrix} \qquad M(z) = \begin{pmatrix} 0 & 1 \\ Q(z) & 0 \end{pmatrix} . \end{aligned} \tag{3.16}$$

It follows from (3.16) that

$$V(z) = -\tfrac{1}{2}\dot{U}(z) \qquad W(z) = -\tfrac{1}{2}\ddot{U}(z) + U(z)Q(z) \qquad \dot{W}(z) = 2V(z)Q(z)\,. \tag{3.17}$$

To construct the Lax representation we have to define $U(z)$ and $Q(z)$. Let us introduce the following ansatz:

$$U(z) = \prod_{i=1}^{N}(z - X_i) \qquad Q(z) = \zeta + 2\tilde{\wp}(8it) \tag{3.18}$$

where the polynomials are $U(z)$ and the function $\zeta = \zeta(z)$ is the expression for the second cover taken from table A3 and the quantity $\tilde{\wp}(8it)$ is expressed in terms of X_i from (3.13)–(3.15) with the help of the Viett theorem.

The spectral curve has the form

$$Y^2 = w^2(z)\left(\frac{\partial\tilde{\wp}}{\partial z}\right)^2 \tag{3.19}$$

where the polynomial w^2 is taken from table A1 and $\tilde{\wp}$ is the rational function taken from table A3.

To find these Lax representations we use the Lax representation for the dynamics associated with the curve $\tilde{C}_1$, with

$$U(\zeta) = \zeta - \tilde{\wp}(8it) \qquad Q(\zeta) = \zeta + 2\tilde{\wp}(8it) \tag{3.20}$$

and raise this representation to the curve C_2 using the formulae for the cover.

For example, let us consider the particle dynamics associated with the two-gap Lamé potential u_3 which is described by the equations

$$\wp_{12}' + \wp_{13}' = 0 \qquad \wp_{21}' + \wp_{23}' = 0 \qquad \wp_{31}' + \wp_{32}' = 0 \tag{3.21}$$

and

$$\dot{x}_1 = -12\wp_{23} \qquad \dot{x}_2 = -12\wp_{13} \qquad \dot{x}_3 = -12\wp_{12}\,. \tag{3.22}$$

The entries U and Q to the matrices L and M have the form

$$U(z) = 4(z - X_1)(z - X_2)(z - X_3) \tag{3.23}$$

$$Q(z) = 4z^3 - 9g_2z + 8X_1X_2X_3 + 27g_3 \tag{3.24}$$

where, in this case, $X_i = 3\wp_{jk}$. The curve $\det(L(z) - yI) = 0$ has the form

$$y^2 = \tfrac{1}{16}(4z^2 - 3g_2)^2(z^2 - 3g_2)(4z^3 - 9g_2z + 27g_3)\,. \tag{3.25}$$

The Lax representations allow to construct the linear r-matrix algebra of the form (2.4) which we will discuss elsewhere.

4. Elliptic potentials from the theta functional point of view

Let $C_g = (w, z)$ be hyperelliptic non-singular curve of genus g, define by the equation

$$w^2 = \prod_{j=1}^{2g+1}(z - z(Q_j)),\ z(Q_j) = z_j \in C_g \qquad z_i \neq z_j \tag{4.1}$$

and realized by means of the function z as a two-sheeted covering of the Riemann sphere with branching points at $Q_1, \ldots, Q_{2g+1}, Q_{2g+2}, z(Q_{2g+2}) = \infty$.

Let us fix on (4.1) the homology basis $(A, B) = (A_1, \ldots, A_g; B_1, \ldots, B_g)$ on C_g and a canonically conjugated basis of holomorphic differentials $v = (v_1, \ldots, v_g)$ in such a way that the Riemann matrix has the form

$$\left(\oint_{A_1} v, \ldots, \oint_{A_g} v; \oint_{B_1} v, \ldots, \oint_{B_g} v\right) = (1_g; \tau)$$

with the matrix τ belonging to Siegel upper half space S_g of degree g. Let us denote by $A(Q) = \int_\infty^Q v$ the Abel map $C_g \to J(C_g)$, where $J(C_g)$ is the Jacobian of the curve C_g.

Let us determine the Riemann theta function $\theta[\varepsilon](z|\tau)$ on $\mathbb{C}^g \times S_g$ with the characteristics

$$[\varepsilon] = \begin{bmatrix} \varepsilon' \\ \varepsilon'' \end{bmatrix} = \begin{bmatrix} \varepsilon'_1 & \ldots & \varepsilon'_g \\ \varepsilon''_1 & \ldots & \varepsilon''_g \end{bmatrix}$$

by the formula

$$\theta[\varepsilon](z|\tau) = \sum_{m \in \mathbb{Z}^g} \exp \pi \mathrm{i} \left\{ \left\langle (m + \tfrac{1}{2}\varepsilon')\tau, (m + \tfrac{1}{2}\varepsilon') \right\rangle + 2\left\langle (m + \tfrac{1}{2}\varepsilon'), z + \tfrac{1}{2}\varepsilon'' \right\rangle \right\} \tag{4.2}$$

where $\langle \cdot, \cdot \rangle$ means the Euclidean scalar product. For integer characteristics we have

$$\theta \begin{bmatrix} \varepsilon' \\ \varepsilon'' \end{bmatrix} (z|\tau) = \exp \pi \, \mathrm{i} \left[\tfrac{1}{4} \langle \varepsilon', \tau \varepsilon' \rangle + \langle \varepsilon', z \rangle + \tfrac{1}{2} \langle \varepsilon', \varepsilon'' \rangle \right] \theta \left(z + \tfrac{1}{2} I \varepsilon'' + \tfrac{1}{2} \tau \varepsilon' | \tau \right) . \tag{4.3}$$

If ε'_i and ε''_j are equal to 0 or 1, the characteristics $[\varepsilon]$ are the characteristics of the half-periods. The theta function (4.2) is odd or even if $[\varepsilon]$ is a half-period characteristic, and we call the corresponding $[\varepsilon]$ odd or even.

The function (4.2) satisfies the two sets of functional equations (see, for example, [21]), the *transformational property*

$$\theta[\varepsilon](z + n'' + \tau n' | \tau) = \exp \pi \mathrm{i} \left[-\langle n' \tau, n' \rangle - 2\langle n', z \rangle - \langle \varepsilon', n' \rangle + \langle \varepsilon'', n' \rangle \right] \theta[\varepsilon](z|\tau) \tag{4.4}$$

where $n', n'' \in \mathbb{Z}^g$ and the *modular property*, which describes the transformation of the theta function under the action of the group $Sp_{2g}(\mathbb{Z})$.

The almost-periodic function $u(x)$ is called a *finite-gap potential* if the spectrum of the Schrödinger operator $H = -\partial_x^2 + u(x)$ is a union of the finite set of segments of a Lebesque (double absolutely continuous) spectrum. Let us formulate the Its–Matveev theorem [16].

Theorem 1 (Its–Matveev theorem). The potential $u(x)$ and the eigenfunction $\Psi(Q, x)$ of the Schrödinger operator $H = -\partial_x^2 + u(x)$ associated with the g-gap Lebesque spectrum $\Sigma = [z_1 z_2] \cup [z_3, z_4] \cup \ldots \cup [z_{2g+1}, \infty]$, are expressed by the formula

$$u(x) = -2 \frac{\partial^2}{\partial x^2} \ln \theta(\mathrm{i} \boldsymbol{U} x - \boldsymbol{A}(\mathcal{D}) - \boldsymbol{K} | \tau) + C \tag{4.5}$$

$$\Psi(Q, x) = \frac{\theta(\mathrm{i} \boldsymbol{U} x + \boldsymbol{A}(Q) - \boldsymbol{A}(\mathcal{D}) - \boldsymbol{K} | \tau)}{\theta(\mathrm{i} \boldsymbol{U} x - \boldsymbol{A}(\mathcal{D}) - \boldsymbol{K} | \tau)} \exp \left(\mathrm{i} x \int_\infty^Q \Omega \right) . \tag{4.6}$$

Here Q is a point on a hyperelliptic Riemann curve (4.1). Ω is the differential of the second kind with zero A-periods which has a second-order pole at infinity with the principal part $\xi^{-2} \mathrm{d}\xi$, where ξ is a local coordinate, $\boldsymbol{U}$ is the vector of B-periods of the differential Ω, $\mathcal{D}$ is a non-special divisor, $\boldsymbol{K}$ is the vector of Riemann constants.

The components U_i, $i = 1, \ldots, g$ of the *winding vector* $\boldsymbol{U}$ in (4.5) and (4.6) are expressed in terms of the normalizing constants c_{ij} of the holomorphic differentials and projections of the branching points $z_1, \ldots, z_{2g+1}$ by the formulae

$$U_j = -2\mathrm{i} c_{1j} \qquad j = 1, \ldots, g . \tag{4.7}$$

Further, we shall restrict ourselves the case of genus-2 curves.

Let us give the theta functional construction of the two-gap elliptic potentials. Following section 2, we describe such points $\tau \in \mathcal{S}_2$, for which the function (4.5) is elliptic. For this purpose we consider the *Humbert surface* H_Δ, $\Delta = N^2$, i.e. the variety

$$H_\Delta = \big\{\alpha\tau_{11} + \beta\tau_{12} + \gamma\tau_{22} + \delta\big(\tau_{12}^2 - \tau_{11}\tau_{22}\big) + \varepsilon = 0 \\ \alpha, \beta, \gamma, \delta, \varepsilon \in \mathbb{Z}, \quad \Delta = \beta^2 - 4(\alpha\gamma + \varepsilon\delta)\big\}. \tag{4.8}$$

The quantity Δ is an invariant with respect to the action of the group $Sp_4(\mathbb{Z})$ [18]. The following theorem summarizes the Weierstrass reduction theory for the case of genus-2 algebraic curves (see, for example, [18, 14]).

Theorem 2 (Reduction theorem). Let C_2 and C_1 be the curves of genus 2 and 1, which are equipped by the homology basis $(A_1, A_2; B_1, B_2)$ and (A, B). The curve C_2 is an N-sheeted covering of the torus C_1 if and only if the moduli of C_2 belong to the Humbert surface with $\Delta = N^2$ and the integer numbers $\alpha, \beta, \gamma, \delta, \varepsilon$, being expressed in terms of of the elements of the integer matrix M, mapping the basis (A_1, A_2, B_1, B_2) into the basis (A, B)

$$M \begin{pmatrix} A_1 \\ A_2 \\ B_1 \\ B_2 \end{pmatrix} = \begin{pmatrix} A \\ B \end{pmatrix} \qquad M = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \\ m_{31} & m_{32} \\ m_{41} & m_{42} \end{pmatrix}$$

are given by the following formulae:

$$\begin{aligned} &\alpha = m_{12}m_{41} - m_{12}m_{42} && \gamma = m_{21}m_{32} - m_{31}m_{22} \\ &\delta = m_{12}m_{21} - m_{11}m_{22} && \varepsilon = m_{31}m_{42} - m_{41}m_{32} \\ &\beta = m_{11}m_{32} - m_{31}m_{12} - (m_{21}m_{42} - m_{41}m_{22}). \end{aligned} \tag{4.9}$$

Moreover, there exists an element $\sigma \in Sp_4(\mathbb{Z})$ and a point $\tau \in \mathcal{S}_2$ such that

$$\sigma \circ \tau = \begin{pmatrix} \tau_{11} & 1/N \\ 1/N & \tau_{22} \end{pmatrix}. \tag{4.10}$$

Under the conditions of the reduction theorem, the two-dimensional theta function is reduced with the help of the addition theorem for theta functions of Nth order (see, for example, [17]) to the finite sum of products of Jacobian theta functions with the moduli $N\tau_{11}$ and $N\tau_{22}$.

Below we apply the Weierstrass reduction theory to describe all elliptic genus-2 potentials.

Lemma 1. The function

$$f(x) = -2\partial_x^2 \ln\theta\left(\frac{x}{2\omega} + \alpha, \beta \left| \frac{1}{N} \begin{pmatrix} \omega'/\omega & 1 \\ 1 & \tilde{\tau} \end{pmatrix} \right.\right) \tag{4.11}$$

with arbitrary $(\alpha, \beta) \in J(C_2)$, $\operatorname{Im}\omega'/\omega = \operatorname{Im}\tau > 0$ is an Nth order elliptic function with primitive periods 2ω, $2\omega'$ and can be represented in the form

$$f(x) = 2\sum_{j=1}^{n} \wp(x - x_j) + 6\sum_{k=1}^{m} \wp(x - x_k). \qquad n + 3m = N \tag{4.12}$$

with x_j belonging to the locus $\mathcal{L}_N$ or its closure.

Proof. It follows from the transformational properties of theta functions that the function (4.11) is a doubly periodical function on the torus C_1 with the primitive periods 2ω, $2\omega'$ and $\tau = \omega'/\omega$. Let us calculate the number of poles of the (4.11). To do this we consider the function

$$g(x) = \frac{\theta\left(x/2\omega + \alpha, \beta | (1/N)\begin{pmatrix} \omega'/\omega & 1 \\ 1 & \tau' \end{pmatrix}\right)}{\vartheta_3\left(Nx/2\omega + N\alpha | \tau\right)} \tag{4.13}$$

where ϑ_3 is a Jacobi theta function. The function $f(x)$ is meromorphic on the torus C_1 as follows from the transformational properties of the theta function. Further, the denominator (4.13) has exactly N zeros in C_1:

$$x = \frac{2k+1}{N}\omega' + \omega - 2\alpha\omega \qquad k = 0, 1, \ldots, N-1 .$$

Therefore, according to the Abel theorem, the numerator has exactly N zeros. These are $x_1, \ldots, x_N$. To prove that the function (4.11) can be written in the form (4.12) we note that the function (4.11) is a two-gap potential and therefore the corresponding wavefunction can have a pole of no more than second order. Using the Schrödinger equation we find the coefficients 2 and 6 in the decomposition (4.12). The proof that the points $x_1, \ldots, x_N$ belong to the locus $\mathcal{L}_N$ is carried out by substituting the ansatz (4.12) into the Schrödinger equation and equating the principal parts of poles to zero.

Theorem 3 (Main theorem). The two-gap potential as defined by (4.5) is an elliptic function of the Nth order if and only if

(i) C_2 covers a torus C_1 N-sheetedly;
(ii) $U_1 U_2 = 0$.

Proof.
Sufficiency. Suppose the conditions of the theorem are fulfilled and for definiteness $U_1 = 0$. Then the function (4.5) is an elliptic function of order N according to the lemma.

Necessity. Let us suppose that the potential (4.5) is an elliptic function with periods $2\omega, 2\omega', \operatorname{Im}\omega/\omega' > 0$. Then the following identities are valid due to the transformation properties of the theta function (4.4):

$$\begin{aligned} 2U_1\omega &= r(n+p'\tau_{11}+q'\tau_{12}) \qquad & 2U_2\omega &= r(m+p'\tau_{12}+q'\tau_{22}) \\ 2U_1\omega' &= s(n'+p\tau_{11}+q\tau_{12}) \qquad & 2U_2\omega' &= s(m'+p\tau_{12}+q\tau_{22}) \end{aligned} \tag{4.14}$$

where $n, m, n', m', p, q, p', q' \in \mathbb{Z}$, $r, s \in \mathbb{N}$. Eliminating $U_i\omega'$, $U_i\omega$, $i = 1, 2$ from (4.14) we find that τ belongs to the Humbert surface H_Δ, with

$$\begin{aligned} &\alpha = m'p' - mp, \ \delta = pq' - p'q \\ &\gamma = nq - n'q', \ \varepsilon = nm' - mn' \\ &\beta = np - m'q' - (mq - n'p') . \end{aligned} \tag{4.15}$$

Calculating the invariant Δ, defined in (4.8), we find that $\Delta = N^2$ with $N = np + mq - m'q' - n'p'$. Therefore the assumption of the theorem leads to the conclusion that C_2 covers a torus N-sheetedly. But in this case we can define a matrix M which maps the homology basis on C_2 to the homology basis on C_1. Taking into account (4.9) we find

$$M = \begin{pmatrix} p & -p' \\ q & -q' \\ -n' & n \\ -m' & m \end{pmatrix} \qquad np+mq-m'q'-n'p' = N . \tag{4.16}$$

According to the reduction theorem there exists a transformation $\sigma \in Sp_4(\mathbb{Z})$ which maps the matrix τ to the form (4.10). Therefore we have in the new homology basis

$$\begin{array}{llll} p = N & q = 0 & n = 1 & m = 0 \\ p' = 0 & q' = 0 & n' = 0 & m' = -1\,. \end{array} \tag{4.17}$$

From (4.14) and (4.16) we conclude, that

$$2U_1\omega = r \qquad U_1\omega' = sN\tau_{11} \qquad U_2 = 0 \tag{4.18}$$

and the theorem is proved.

It follows from the conditions of the theorem that elliptic potentials are singled out from finite-gap potentials by some subvariety in the Humbert variety. We shall call this variety *E_Δ-variety*, $E_\Delta \in H_\Delta$.

Let us derive the two-gap Lamé and Treibich–Verdier potentials from the reduction technique of finite gap potentials to elliptic potential developed above.

Proposition 1 (Proposition on the Treibich–Verdier potentials). The only two-gap primitive Lamé and Treibich–Verdier potentials are the three first elliptic two-gap elliptic potentials from table 1.

Proof. Let us consider the elliptic potential

$$u(x) = -2\partial_x^2 \ln\theta[\delta]\left(\frac{x}{2\omega}, 0\left|\begin{pmatrix} \frac{1}{N}\frac{\omega'}{\omega} & \frac{1}{N} \\ \frac{1}{N} & \frac{1}{N}\frac{\tilde{\omega}'}{\tilde{\omega}} \end{pmatrix}\right.\right) \qquad U_2 = 0 \tag{4.19}$$

with $[\delta]$ running through all the six odd characteristics. Let us consider the function

$$\Theta(x) = \theta[\delta]\left(\frac{x}{2\omega}\left|\begin{pmatrix} \frac{1}{N}\frac{\omega'}{\omega} & \frac{1}{N} \\ \frac{1}{N} & \frac{1}{N}\frac{\tilde{\omega}'}{\tilde{\omega}} \end{pmatrix}\right.\right).$$

At $x = 0$ this is a theta constant with the characteristic $[\delta]$. One can calculate, using (4.3), that at $x = \omega$ the characteristics become $[\delta] + \left[\begin{smallmatrix} 00 \\ 10 \end{smallmatrix}\right]$, at $x = \omega'$ the charteristic $[\delta]$ turns into the characteristic $[\delta] + \left[\begin{smallmatrix} N\,N \\ 0\,1 \end{smallmatrix}\right]$ and at $x = \omega + \omega'$ it is $[\delta] + \left[\begin{smallmatrix} N\,N \\ 1\,1 \end{smallmatrix}\right]$. Let us denote by (n_0, n_1, n_2, n_3) the coefficients in the decomposition $u_N = n_0\wp(x) + n_1\wp(x+\omega) + n_2\wp(x+\omega+\omega') + n_3\wp(\omega')$ with $\sum_{k=0}^{3} n_k = N$ and $n_i = 2$ or 6. Let the characteristic $[\delta]$ run through all the odd characteristics. Then for odd N we have

$x = 0$	$x = \omega$	$x = \omega'$	$x = \omega' + \omega'$	(n_0, n_1, n_2, n_3)
$\left[\begin{smallmatrix} 10 \\ 10 \end{smallmatrix}\right]$	$\left[\begin{smallmatrix} 10 \\ 00 \end{smallmatrix}\right]$	$\left[\begin{smallmatrix} 01 \\ 11 \end{smallmatrix}\right]$	$\left[\begin{smallmatrix} 01 \\ 01 \end{smallmatrix}\right]$	$(n_0, 0, n_2, n_3)$
$\left[\begin{smallmatrix} 10 \\ 11 \end{smallmatrix}\right]$	$\left[\begin{smallmatrix} 10 \\ 01 \end{smallmatrix}\right]$	$\left[\begin{smallmatrix} 01 \\ 10 \end{smallmatrix}\right]$	$\left[\begin{smallmatrix} 01 \\ 00 \end{smallmatrix}\right]$	$(n_0, 0, 0, 0)$
$\left[\begin{smallmatrix} 11 \\ 10 \end{smallmatrix}\right]$	$\left[\begin{smallmatrix} 10 \\ 01 \end{smallmatrix}\right]$	$\left[\begin{smallmatrix} 00 \\ 11 \end{smallmatrix}\right]$	$\left[\begin{smallmatrix} 00 \\ 01 \end{smallmatrix}\right]$	$(n_0, 0, 0, 0)$
$\left[\begin{smallmatrix} 01 \\ 01 \end{smallmatrix}\right]$	$\left[\begin{smallmatrix} 01 \\ 11 \end{smallmatrix}\right]$	$\left[\begin{smallmatrix} 10 \\ 00 \end{smallmatrix}\right]$	$\left[\begin{smallmatrix} 10 \\ 10 \end{smallmatrix}\right]$	$(n_0, n_1, 0, n_3)$
$\left[\begin{smallmatrix} 01 \\ 11 \end{smallmatrix}\right]$	$\left[\begin{smallmatrix} 01 \\ 01 \end{smallmatrix}\right]$	$\left[\begin{smallmatrix} 10 \\ 10 \end{smallmatrix}\right]$	$\left[\begin{smallmatrix} 10 \\ 00 \end{smallmatrix}\right]$	$(n_0, n_1, n_2, 0)$
$\left[\begin{smallmatrix} 11 \\ 01 \end{smallmatrix}\right]$	$\left[\begin{smallmatrix} 11 \\ 11 \end{smallmatrix}\right]$	$\left[\begin{smallmatrix} 00 \\ 00 \end{smallmatrix}\right]$	$\left[\begin{smallmatrix} 00 \\ 10 \end{smallmatrix}\right]$	$(n_0, 0, 0, 0)$.

We see that the only possibilities are $u(x) = 6\wp(x)$ or $2\wp(x+\omega_1) + 2\wp(x+\omega_2) + 2\wp(x+\omega_3)$ and $u(x) = 6u(x) + 2\wp(x+\omega_i) + 2\wp(x+\omega_k)$.

For even N we have

$$\begin{array}{lllll}
x=0 & x=\omega & x=\omega' & x=\omega'+\omega' & (n_0,n_1,n_2,n_3) \\
\left[\begin{smallmatrix}10\\10\end{smallmatrix}\right] & \left[\begin{smallmatrix}10\\00\end{smallmatrix}\right] & \left[\begin{smallmatrix}10\\11\end{smallmatrix}\right] & \left[\begin{smallmatrix}10\\01\end{smallmatrix}\right] & (n_0,0,n_2,0) \\
\left[\begin{smallmatrix}10\\11\end{smallmatrix}\right] & \left[\begin{smallmatrix}10\\01\end{smallmatrix}\right] & \left[\begin{smallmatrix}10\\10\end{smallmatrix}\right] & \left[\begin{smallmatrix}10\\00\end{smallmatrix}\right] & (n_0,0,n_2,0) \\
\left[\begin{smallmatrix}11\\10\end{smallmatrix}\right] & \left[\begin{smallmatrix}11\\00\end{smallmatrix}\right] & \left[\begin{smallmatrix}11\\11\end{smallmatrix}\right] & \left[\begin{smallmatrix}11\\01\end{smallmatrix}\right] & (n_0,0,0,n_3) \\
\left[\begin{smallmatrix}01\\01\end{smallmatrix}\right] & \left[\begin{smallmatrix}01\\11\end{smallmatrix}\right] & \left[\begin{smallmatrix}01\\00\end{smallmatrix}\right] & \left[\begin{smallmatrix}01\\10\end{smallmatrix}\right] & (n_0,n_1,0,0) \\
\left[\begin{smallmatrix}01\\11\end{smallmatrix}\right] & \left[\begin{smallmatrix}01\\01\end{smallmatrix}\right] & \left[\begin{smallmatrix}01\\10\end{smallmatrix}\right] & \left[\begin{smallmatrix}01\\00\end{smallmatrix}\right] & (n_0,n_1,0,0) \\
\left[\begin{smallmatrix}11\\01\end{smallmatrix}\right] & \left[\begin{smallmatrix}11\\11\end{smallmatrix}\right] & \left[\begin{smallmatrix}11\\00\end{smallmatrix}\right] & \left[\begin{smallmatrix}11\\10\end{smallmatrix}\right] & (n_0,0,0,n_3)\,.
\end{array}$$

We see that the only possibilities in this case are $u(x) = 6\wp(x) + 2\wp(x+\omega_i)$. The proposition is proved.

4.1. *Elliptic subvarieties of Humbert surfaces*

The components of the Humbert surface are described in terms of the vanishing of some modular forms [18], more generally, the Humbert surface H_Δ is described by some ideal in the ring of modular forms [29]. Therefore it is natural to describe elliptic subsurfaces E_Δ of H_Δ, $\Delta = N^2$ in terms of the vanishing of some theta constants.

Proposition 2 (Proposition on the elliptic points). Let the non-singular curve associated with the two-gap potential cover a torus N-sheetedly. Let us fix such a homology basis that the matrix τ has the form (4.10) and belongs to the component H_Δ. Then elliptic points in H_Δ are separated by the condition

$$\theta_i[\delta]\left(0\middle|\begin{pmatrix} \tau & 1/N \\ 1/N & \tilde{\tau}\end{pmatrix}\right) = 0 \tag{4.20}$$

where $[\delta]$ runs through all the six odd characteristics and $i = 1$ or 2.

Proof. It follows immediately from the *Rosenhain formulae for the normalizing constants of the holomorphic differentials*† and the expression (4.7) for the winding vectors. Assume that the curve C_2 has the form $w^2 = z(z-1)(z-\lambda_1)(z-\lambda_2)(z-\lambda_3)$; then the normalizing constants of the holomorphic differentials $v_i = (c_{i1}z+c_{i2})\mathrm{d}z/w$, $i = 1, 2$ have the form [19]

$$c_{i1} = -\frac{\theta_1[\varepsilon_i]}{2\pi^2\theta[\delta_{i1}]\theta[\delta_{i2}]\theta[\delta_{i3}]} \tag{4.21}$$

where $[\varepsilon_i]$ is an odd theta constant and $[\delta_i]$, $i = 1, 2$ and 3 are such even theta constants that $[\varepsilon_i] = [\delta_{i1}] + [\delta_{i2}] + [\delta_{i3}]$.

Let us give a few examples for the condition (4.20). The simplest ones are at $N = 2^p$, $p = 1, \ldots$ because to simplify we can apply the the addition theorem for the theta functions of the second order (see, for example, [21])

$$\Theta[\varepsilon](x|\tau)\theta[\delta](y|\tau) = \sum_\rho \theta\begin{bmatrix}\frac{1}{2}(\varepsilon'+\delta')+\rho \\ \varepsilon''+\delta''\end{bmatrix}(x+y|2\tau)\theta\begin{bmatrix}\frac{1}{2}(\varepsilon'-\delta')+\rho \\ \varepsilon''-\delta''\end{bmatrix}(x-y|2\tau) \tag{4.22}$$

where the summation runs over $\rho = (0,0), (0,1), (1,0), (1,1)$. The particular cases of (4.22) which are necessary for the calculations are given in appendix B. Below we also use

† These formulae are a consequence of the important *Rosenhain derivative formulae* given in appendix B.

the formula

$$\Theta\begin{bmatrix} \varepsilon_1' & \varepsilon_2' \\ \varepsilon_1'' & \varepsilon_2'' \end{bmatrix}\left(z\middle|\begin{pmatrix} \tau & 1 \\ 1 & \widetilde{\tau} \end{pmatrix}\right) = \mathrm{e}^{-\frac{1}{2}\pi i\varepsilon_1'\varepsilon_2'}\theta\begin{bmatrix} \varepsilon_1' & \varepsilon_2' \\ \varepsilon_1''+\varepsilon_2' & \varepsilon_2''+\varepsilon_1' \end{bmatrix}\left(z\middle|\begin{pmatrix} \tau & 0 \\ 0 & \widetilde{\tau} \end{pmatrix}\right). \tag{4.23}$$

For example, the condition (4.20) for $N = 2$ is $\vartheta_1'\vartheta_3\widetilde{\vartheta}_4^2 = 0$, where $\vartheta_i = \vartheta_i(0|2\tau)$, $\widetilde{\vartheta}_i = \vartheta_i(0|2\widetilde{\tau})$. This condition is not satisfied for non-singular tori. Therefore elliptic potentials of the form $2\wp(x-x_1)+2\wp(x-x_2)$ do not exist.

Example $N = 4$. For $N = 4$ the condition (4.20) written for the characteristic $\left[\begin{smallmatrix}11\\01\end{smallmatrix}\right]$ reads

$$\frac{\sqrt{2}\vartheta_2^2\widetilde{\vartheta}_3}{\vartheta_3} + \sqrt{\vartheta_3^2\widetilde{\vartheta}_3^2 + \vartheta_2^2\widetilde{\vartheta}_4^2 - \vartheta_4^2\widetilde{\vartheta}_2^2} = 0 \tag{4.24}$$

where $\vartheta_i = \vartheta_i(0|4\tau)$, $\widetilde{\vartheta}_i = \vartheta_i(0|4\widetilde{\tau})$. To obtain (4.24), we used (4.22) twice. The condition (4.24) rewritten in terms of the Jacobi moduli $k = \vartheta_2^2/\vartheta_3^2$, $\widetilde{k} = \widetilde{\vartheta}_2^2/\widetilde{\vartheta}_3^2$, coincides with those given in table A4.

The condition (4.24) is equivalent to the relations between Jacobi theta constants

$$\tilde{\vartheta}_2^2 = \tilde{\vartheta}_3^2\frac{\vartheta_4^2}{\vartheta_3^2}\left(1-4\frac{\vartheta_2^4}{\vartheta_3^4}\right) \qquad \tilde{\vartheta}_4^2 = \tilde{\vartheta}_3^2\frac{\vartheta_2^2}{\vartheta_3^2}\left(1-4\frac{\vartheta_4^4}{\vartheta_3^4}\right). \tag{4.25}$$

Let us derive the potential u_4 by direct computation. We have according to theorems 1 and 2

$$\begin{aligned} u_4(x) &= -2\partial_x^2 \ln\Theta(x) + C \\ \Theta(x) &= \theta[\delta]\left(\frac{x}{2\omega},0\middle|\begin{pmatrix} \tau & \frac{1}{4} \\ \frac{1}{4} & \widetilde{\tau} \end{pmatrix}\right). \end{aligned} \tag{4.26}$$

Let us consider the definite case $[\delta] = \left[\begin{smallmatrix}11\\01\end{smallmatrix}\right]$. Applying (4.22) twice we have

$$\begin{aligned} \Theta(x) = {}& 8\frac{\left(\widehat{\widehat{\theta}}\left[\begin{smallmatrix}10\\00\end{smallmatrix}\right](z)\widehat{\widehat{\theta}}\left[\begin{smallmatrix}00\\00\end{smallmatrix}\right](z) - \widehat{\widehat{\theta}}\left[\begin{smallmatrix}01\\00\end{smallmatrix}\right](z)\widehat{\widehat{\theta}}\left[\begin{smallmatrix}11\\00\end{smallmatrix}\right](z)\right)}{\widehat{\theta}\left[\begin{smallmatrix}01\\10\end{smallmatrix}\right]\widehat{\theta}\left[\begin{smallmatrix}01\\00\end{smallmatrix}\right]} \\ &\times\left(\widehat{\widehat{\theta}}\left[\begin{smallmatrix}01\\01\end{smallmatrix}\right](z)\widehat{\widehat{\theta}}\left[\begin{smallmatrix}00\\01\end{smallmatrix}\right](z) + \widehat{\widehat{\theta}}\left[\begin{smallmatrix}10\\01\end{smallmatrix}\right](z)\widehat{\widehat{\theta}}\left[\begin{smallmatrix}11\\01\end{smallmatrix}\right](z)\right) \\ &+4\frac{\left(\widehat{\widehat{\theta}}\left[\begin{smallmatrix}10\\01\end{smallmatrix}\right](z)\widehat{\widehat{\theta}}\left[\begin{smallmatrix}01\\01\end{smallmatrix}\right](z) + \widehat{\widehat{\theta}}\left[\begin{smallmatrix}11\\01\end{smallmatrix}\right](z)\widehat{\widehat{\theta}}\left[\begin{smallmatrix}00\\01\end{smallmatrix}\right](z)\right)}{\widehat{\theta}\left[\begin{smallmatrix}11\\00\end{smallmatrix}\right]\widehat{\theta}\left[\begin{smallmatrix}00\\01\end{smallmatrix}\right]} \\ &\times\left(\widehat{\widehat{\theta}}^2\left[\begin{smallmatrix}00\\00\end{smallmatrix}\right](z) - \widehat{\widehat{\theta}}^2\left[\begin{smallmatrix}11\\00\end{smallmatrix}\right](z) + \widehat{\widehat{\theta}}^2\left[\begin{smallmatrix}10\\00\end{smallmatrix}\right](z) - \widehat{\widehat{\theta}}^2\left[\begin{smallmatrix}01\\00\end{smallmatrix}\right](z)\right) \end{aligned} \tag{4.27}$$

where we denote $\widehat{\theta}[\varepsilon] = \theta[\varepsilon](0|2\tau)$, $\widehat{\widehat{\theta}}[\varepsilon](z) = \theta[\varepsilon](z|4\tau)$ and $z = (x/2\omega, 0)$.

Using (4.23), we rewrite (4.27) in the form

$$\begin{aligned} \Theta(x) = {}& 4i\frac{\vartheta_1(x/2\omega)\vartheta_3(x/2\omega)\widetilde{\vartheta}_2\vartheta_4}{\sqrt{\vartheta_2\vartheta_4\widetilde{\vartheta}_3\widetilde{\vartheta}_4}} \\ &\times\left\{\frac{\sqrt{2}\vartheta_2^2(x/2\omega)\widetilde{\vartheta}_3}{\vartheta_3} + \frac{\vartheta_3^2(x/2\omega)\widetilde{\vartheta}_3^2 + \vartheta_2^2(x/2\omega)\widetilde{\vartheta}_4^2 - \vartheta_4^2(x/2\omega)\widetilde{\vartheta}_2^2}{\sqrt{\vartheta_3^2\widetilde{\vartheta}_3^2 + \vartheta_2^2\widetilde{\vartheta}_4^2 - \vartheta_4^2\widetilde{\vartheta}_2^2}}\right\}. \end{aligned} \tag{4.28}$$

By the conditions (4.24), (4.25) and relations between the squares of Jacobi theta functions [3], one can prove that $\Theta(x)$ is proportional to $\vartheta_1^3(x/2\omega)\vartheta_3(x/2\omega)$ and therefore the potential u_4 has the form given in table 1.

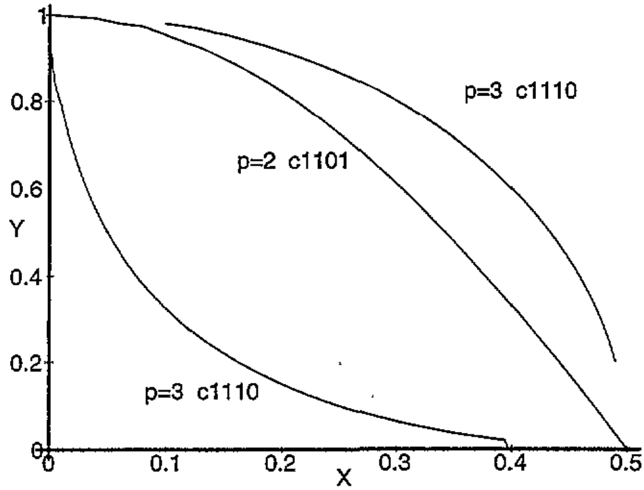

Figure 1. Components of E_4 and E_8.

Further examples. Let us consider the function

$$u(x) = -2\partial_x^2 \ln \theta[\delta] \left(\frac{x}{2\omega}, 0 \middle| \begin{pmatrix} \tau & 1/2^p \\ 1/2^p & \tilde{\tau} \end{pmatrix} \right) \tag{4.29}$$

with $p > 2$ and the moduli τ and $\tilde{\tau}$ are connected by the condition

$$\theta_i[\delta] \left(0 \middle| \begin{pmatrix} \tau & 1/2^p \\ 1/2^p & \tilde{\tau} \end{pmatrix} \right) = 0 . \tag{4.30}$$

One can show (see the formulae for theta constants of the 2^p-sheeted cover) that (4.30) is valid and E_{2^p} is not empty for $p = 2, 3, \ldots$. In particular, denoting $X = \mathrm{i}\vartheta_2(0; 2^p\tau)/\vartheta_3(0; 2^p\tau)$, $Y = \vartheta_2(0; 2^p\tilde{\tau})/\vartheta_3(0; 2^p\tilde{\tau})$ we plot below two varieties E_{2^p} for $p = 2$ and $p = 3$, respectively, in the coordinates X and Y.

The curve shown in figure 1 corresponds to a family of elliptic potentials. We emphasize that the potential u_8 is a new elliptic potential connected with an eight-sheeted cover of the torus. It differs from the u_8 Treibich–Verdier potential which is not primitive and can be obtain from the Treibich–Verdier potential u_4 by the Gauss transform (3.1) of the moduli of one of the tori.

We can summarize all the discussion by the following statement.

Conjecture 1. There exist infinitely many primitive elliptic potentials $u_N(x)$ of genus-2 at $N \in \mathbb{N}$. Therefore the two gap locus of Calogero–Moser system has infinitely many components.

The elliptic potentials exist at $N = 4, 5$ and 8. To prove the conjecture it is sufficient to find solutions for (4.20) for a countable number of N.

Acknowledgments

The authors are grateful to E D Belokolos for many suggestions for reduction techniques for finite-gap potentials and to H W Braden, A P Fordy, V B Kuznetsov, E K Sklyanin and T Suzuki for valuable discussions. VZE would like to acknowledge the support of the Royal Society within ex-quota grant and the support of the International Science Foundation within grant no U44000.

Appendix A. Two-gap Lamé and Treibich–Verdier potentials

Tables A1–A5 give the complete description of two primitive Treibich–Verdier potential which includes the explicit formulae for the covers, the link between moduli of the tori, wavefunctions and Lamé polynomials. We also give for complicity the analogous description of the two-gap Lamé potential, which is known. We note that some of these results concerning Treibich–Verdier potentials were first given in [6, 25, 10].

Table A1. The spectral curves.

u_N	The spectral curve $C_2 = (w, z)$	The coordinate λ in terms of w, z
u_3	$w^2 = -(z^2 - 3g_2)\prod_{i=1}^{3}(z - 3e_i)$ $\lambda^3 - 3\lambda\wp + \wp' = 0$	$\lambda = \frac{2w}{3z^2-3g_2}$
u_4	$w^2 = -(z + 6e_i)\prod_{k=1}^{4}(z - z_k(i)), i = 1, 2, 3$ $z_{1,2}(i) = e_j + 2e_i \pm 2\sqrt{(e_i - e_j)(2e_j + 7e_i)}$ $z_{3,4}(i) = e_k + 2e_i \pm 2\sqrt{(e_i - e_k)(2e_k + 7e_i)}$ $\lambda^4 - 3(2\wp - e_i)\lambda^2 + 4\lambda\wp' - 3(\wp - e_j)(\wp - e_k) = 0,$ $i \neq j \neq k = 1, 2, 3$	$\lambda = \frac{3w}{2(6e_j+z)(15e_j-2z)}$
u_5	$w^2 = \prod_{i=1}^{i=5}(z - z_i(j)), \quad j = 1, 2, 3$ $z_4(j) = 6e_k - 3e_i,\ z_5(j) = 6e_i - 3e_k$ $\prod_{i=1}^{3}(z - z_i(j))$ $= z^3 - 3z^2e_j + (51e_j^2 - 20g_2)z - 369e_j^3 + 132e_jg_2$ $\lambda^5 - 2(e_i + 5\wp)\lambda^3 + 10\wp\lambda^2 + 3(e_i + 5\wp)(e_i - \wp)\lambda$ $-2\wp'(e_i - \wp) = 0$	$\lambda = \frac{-4w}{5z^2+6ze_j+261e_j^2-108g_2}$

Table A2. The first cover.

u_N	The cover π	The reduction of the holomorphic differential
u_3	$\wp(u) = -\frac{z^3-27g_3}{9(z^2-3g_2)}$	$\frac{d\wp}{\wp'} = -\frac{3z}{2}\frac{dz}{w}$
u_4	$\wp(u) = e_j + \frac{(z-3e_i+9e_k)^2(z-z_1(i))(z-z_2(i))}{4(z+6e_i)(-2z+15e_i)^2}$ $\wp(u) = e_k + \frac{(z-3e_i+9e_j)^2(z-z_3(i))(z-z_4(i))}{4(z+6e_i)(-2z+15e_i)^2}$	$\frac{d\wp}{\wp'} = -(2z + 3e_k)\frac{dz}{w}$
u_5	$\wp(u) = e_j + \frac{(z-z_1(j))(z-z_2(j))(z-z_3(j))(z+15e_j)^2}{(5z^2+6ze_j+261e_j^2-108g_2)^2}$	$\frac{d\wp}{\wp'} = (5z + 3e_j)\frac{dz}{2w}$

Table A3. The second cover.

u_N	The cover $\widetilde{\pi}$
u_3	$\widetilde{\wp}(u) = -\frac{9}{16}(4z^3 - 9g_2z + 9g_3)$
u_4	$\widetilde{\wp}(u) - \widetilde{e}_j = -\frac{9(z-z_2)(z-z_3)(z+4e_i-e_k)^2}{4(z+6e_i)}$
u_5	$\widetilde{\wp}(u) = -\frac{9P_5(z)}{4(z-3e_i-9e_j)(z+6e_i+9e_j)}$ $P_5(z) = z^5 + 3e_i^4z^4 - 42e_i^2z^3 + 150e_je_kz^3 + 30e_i^2e_kz^2$ $+9(69e_i^4 + 625e_j^2e_k^2 - 221e_ig_3/2)z$ $-27e_i(11e_i^4 - 81e_ig_k/2 + 475e_k^2e_j^2)$

Table A4. The link between moduli of the tori C_1 and $\widetilde{C}_1$.

u_3	$\widetilde{g}_2 = \frac{3^7(g_2^3+9g_3^2)}{2^6}, \quad \widetilde{g}_3 = \frac{3^{11}(g_3 g_2^3 - 3g_3^3)}{2^9}$
	$\frac{1}{\widetilde{k}} = \frac{1}{2} + \frac{(k^2-2)(2k^2-1)(k^2+1)}{4(k^4-k^2+1)^{3/2}}$
u_4	$\widetilde{g}_2 = 3^7 \frac{83}{4} g_2 g_3 e_i + 3^6 \frac{89}{4} g_2^2 e_i^2 + 3^4 g_2^2 + 14 \times 3^7 g_3^2,$
	$\widetilde{g}_3 = \frac{1}{4} 3^7 \times 11 \times 17 e_i g_2^4 + \frac{1}{8} 3^{10} \times 307 e_i g_3^2 g_2 + \frac{1}{16} 3^{10} \times 457 g_2^2 g_3 e_i^2 + \frac{1}{4} 3^8 \times 61 g_3 g_2^3 + \frac{1}{16} 3^{11} \times 5 \times 19 g_3^3$
	$\widetilde{k} = k'(1-4k^2)$ or equivalently $\widetilde{k}' = k(1-4(k')^2)$
u_5	$\widetilde{g}_2 = 2^2 \times 3^7 \times 5 \times 23 e_i^6 - 3^7 \times 11 \times 17 e_i^4 g_2 - \frac{3^6 \times 5^3}{2^2} e_i^2 g_2^2 + \frac{3^4}{2^4} 5^5 g_2^3$
	$\widetilde{g}_3 = -\frac{3^7}{2^6} e_i \left(5^7 g_2^4 + 2^8 \times 3^5 \times 191 e_i^8 + 2^5 \times 3^3 \times 13 \times 457 e_i^4 g_2^2 \right.$
	$\left. -2^8 \times 3^3 \times 23 \times 79 e_i^6 g_2 - 2^4 \times 3^2 \times 5^4 \times 11 e_i^2 g_2^3 \right)$

Table A5. The wavefunction and Lamé polynomials.

u_N	The wavefunction $\Psi(x,u)$ and Lamé polynomials $\Lambda(x)$
u_3	$\Psi(x) = \frac{\partial}{\partial x}(\exp(\lambda x)\Phi(x;u))$
	$\Lambda_{ij} = \sqrt{(\wp(x)-e_i)(\wp(x)-e_j)}, (z = 3e_k), \quad i \neq j \neq k = 1,2,3$
	$\Lambda_{\pm} = \wp(x) \pm \frac{1}{2}\sqrt{\frac{g_2}{3}}, (z = \pm\sqrt{3g_2})$
u_4	$\Psi(x,u) = \frac{\partial}{\partial x}(\exp(\lambda x)\Phi(x;u)) + \frac{3\lambda^2 - 3\wp(u) + z}{6\sqrt{\wp(u)-e_i}}\Phi(x+\omega_i;u)\exp(\lambda x)$
	$\Lambda_{ik} = \sqrt{(\wp(x)-e_i)(\wp(x)-e_j)} + \frac{1}{3}[(e_i - e_k) \pm \sqrt{(e_i-e_k)(7e_i+2e_k)}]\sqrt{\frac{\wp(x)-e_j}{\wp(x)-e_i}}$
	$(z = e_k + 2e_i \pm 2\sqrt{(e_i-e_k)(2e_k+7e_i)}), \ k \in \{1,2,3\} - \{i\}$
	$\Lambda_0 = \wp(x) - e_i \quad (z = -6e_i)$
u_5	$\Psi(x,u) = \frac{\partial}{\partial x}(\exp(\lambda x)\Phi(x;u)) + [a_{i,j}\Phi(x+\omega_i,u) + a_{j,i}\Phi(x+\omega_j,u)]\exp(\lambda x)$
	$a_{i,j} = \frac{-9\lambda\wp(u) + \lambda z + 6\lambda e_j + 3\wp'(u)}{6\lambda\sqrt{\wp(u)-e_i} + 6\sqrt{\wp(u)-e_j}\sqrt{\wp(u)-e_k}}$
	$\Lambda_i = \sqrt{(\wp(x)-e_i)(\wp(x)-e_k)} + (e_i - e_j)\sqrt{\frac{\wp(x)-e_k}{\wp(x)-e_i}} \ (z = 6e_i - 3e_j)$
	$\Lambda_j = \sqrt{(\wp(x)-e_j)(\wp(x)-e_k)} + (e_j - e_i)\sqrt{\frac{\wp(x)-e_k}{\wp(x)-e_j}} \ (z = 6e_j - 3e_i)$
	$\Lambda_n = \sqrt{(\wp(x)-e_j)(\wp(x)-e_j)} + \tilde{a}_{ij}\sqrt{\frac{\wp(x)-e_i}{\wp(x)-e_j}} + \tilde{a}_{ji}\sqrt{\frac{\wp(x)-e_j}{\wp(x)-e_i}},$
	$(z = z_k), i \neq j \neq n = 1,2,3, \tilde{a}_{ij} = \frac{15e_i^2 + 27e_j^2 - 6e_ie_j - z_k^2 + 2z_k(e_j - e_i)}{24(e_j - e_i)}$

Appendix B. Theta functional formulae

B.1. Relations between theta constants for $g = 2$

Here we give three groups of formulae which are consequence of the Riemann theta formula for theta constants when $g = 2$. These are the relations between the fourth powers of even theta constants, the relations between the squares of even theta constants and the Rosenhain derivative formulae:

$$\theta^4\begin{bmatrix}00\\00\end{bmatrix} - \theta^4\begin{bmatrix}00\\11\end{bmatrix} = \theta^4\begin{bmatrix}10\\01\end{bmatrix} + \theta^4\begin{bmatrix}01\\00\end{bmatrix} = \theta^4\begin{bmatrix}10\\00\end{bmatrix} + \theta^4\begin{bmatrix}01\\10\end{bmatrix}$$
$$\theta^4\begin{bmatrix}00\\00\end{bmatrix} - \theta^4\begin{bmatrix}11\\00\end{bmatrix} = \theta^4\begin{bmatrix}01\\10\end{bmatrix} + \theta^4\begin{bmatrix}00\\01\end{bmatrix} = \theta^4\begin{bmatrix}00\\10\end{bmatrix} + \theta^4\begin{bmatrix}10\\01\end{bmatrix}$$
$$\theta^4\begin{bmatrix}00\\00\end{bmatrix} - \theta^4\begin{bmatrix}11\\11\end{bmatrix} = \theta^4\begin{bmatrix}10\\00\end{bmatrix} + \theta^4\begin{bmatrix}00\\10\end{bmatrix} = \theta^4\begin{bmatrix}01\\00\end{bmatrix} + \theta^4\begin{bmatrix}00\\01\end{bmatrix}$$

$$
\begin{array}{ll}
\theta^2 \genfrac{[}{]}{0pt}{}{00}{00} \theta^2 \genfrac{[}{]}{0pt}{}{10}{00} = \theta^2 \genfrac{[}{]}{0pt}{}{01}{00} \theta^2 \genfrac{[}{]}{0pt}{}{11}{00} + \theta^2 \genfrac{[}{]}{0pt}{}{00}{01} \theta^2 \genfrac{[}{]}{0pt}{}{10}{01} & \left(\genfrac{[}{]}{0pt}{}{10}{00}\right) \\
\theta^2 \genfrac{[}{]}{0pt}{}{00}{00} \theta^2 \genfrac{[}{]}{0pt}{}{01}{00} = \theta^2 \genfrac{[}{]}{0pt}{}{10}{00} \theta^2 \genfrac{[}{]}{0pt}{}{11}{00} + \theta^2 \genfrac{[}{]}{0pt}{}{01}{10} \theta^2 \genfrac{[}{]}{0pt}{}{00}{10} & \left(\genfrac{[}{]}{0pt}{}{01}{00}\right) \\
\theta^2 \genfrac{[}{]}{0pt}{}{00}{00} \theta^2 \genfrac{[}{]}{0pt}{}{11}{00} = \theta^2 \genfrac{[}{]}{0pt}{}{10}{00} \theta^2 \genfrac{[}{]}{0pt}{}{01}{00} + \theta^2 \genfrac{[}{]}{0pt}{}{00}{11} \theta^2 \genfrac{[}{]}{0pt}{}{11}{11} & \left(\genfrac{[}{]}{0pt}{}{11}{00}\right) \\
\theta^2 \genfrac{[}{]}{0pt}{}{00}{00} \theta^2 \genfrac{[}{]}{0pt}{}{00}{10} = \theta^2 \genfrac{[}{]}{0pt}{}{01}{00} \theta^2 \genfrac{[}{]}{0pt}{}{01}{10} + \theta^2 \genfrac{[}{]}{0pt}{}{00}{01} \theta^2 \genfrac{[}{]}{0pt}{}{00}{11} & \left(\genfrac{[}{]}{0pt}{}{00}{10}\right) \\
\theta^2 \genfrac{[}{]}{0pt}{}{10}{00} \theta^2 \genfrac{[}{]}{0pt}{}{00}{10} = \theta^2 \genfrac{[}{]}{0pt}{}{00}{11} \theta^2 \genfrac{[}{]}{0pt}{}{10}{01} + \theta^2 \genfrac{[}{]}{0pt}{}{01}{10} \theta^2 \genfrac{[}{]}{0pt}{}{11}{00} & \left(\genfrac{[}{]}{0pt}{}{10}{10}\right) \\
\theta^2 \genfrac{[}{]}{0pt}{}{01}{00} \theta^2 \genfrac{[}{]}{0pt}{}{00}{10} = \theta^2 \genfrac{[}{]}{0pt}{}{00}{00} \theta^2 \genfrac{[}{]}{0pt}{}{01}{10} + \theta^2 \genfrac{[}{]}{0pt}{}{11}{11} \theta^2 \genfrac{[}{]}{0pt}{}{10}{01} & \left(\genfrac{[}{]}{0pt}{}{01}{10}\right) \\
\theta^2 \genfrac{[}{]}{0pt}{}{00}{10} \theta^2 \genfrac{[}{]}{0pt}{}{11}{00} = \theta^2 \genfrac{[}{]}{0pt}{}{10}{00} \theta^2 \genfrac{[}{]}{0pt}{}{01}{10} + \theta^2 \genfrac{[}{]}{0pt}{}{11}{11} \theta^2 \genfrac{[}{]}{0pt}{}{00}{01} & \left(\genfrac{[}{]}{0pt}{}{11}{10}\right) \\
\theta^2 \genfrac{[}{]}{0pt}{}{00}{00} \theta^2 \genfrac{[}{]}{0pt}{}{00}{01} = \theta^2 \genfrac{[}{]}{0pt}{}{10}{00} \theta^2 \genfrac{[}{]}{0pt}{}{10}{01} + \theta^2 \genfrac{[}{]}{0pt}{}{00}{11} \theta^2 \genfrac{[}{]}{0pt}{}{00}{10} & \left(\genfrac{[}{]}{0pt}{}{00}{01}\right) \\
\theta^2 \genfrac{[}{]}{0pt}{}{10}{00} \theta^2 \genfrac{[}{]}{0pt}{}{00}{01} = \theta^2 \genfrac{[}{]}{0pt}{}{00}{00} \theta^2 \genfrac{[}{]}{0pt}{}{10}{10} + \theta^2 \genfrac{[}{]}{0pt}{}{11}{11} \theta^2 \genfrac{[}{]}{0pt}{}{01}{10} & \left(\genfrac{[}{]}{0pt}{}{10}{01}\right) \\
\theta^2 \genfrac{[}{]}{0pt}{}{01}{00} \theta^2 \genfrac{[}{]}{0pt}{}{00}{01} = \theta^2 \genfrac{[}{]}{0pt}{}{00}{11} \theta^2 \genfrac{[}{]}{0pt}{}{01}{10} + \theta^2 \genfrac{[}{]}{0pt}{}{10}{01} \theta^2 \genfrac{[}{]}{0pt}{}{11}{00} & \left(\genfrac{[}{]}{0pt}{}{01}{01}\right) \\
\theta^2 \genfrac{[}{]}{0pt}{}{11}{00} \theta^2 \genfrac{[}{]}{0pt}{}{00}{01} = \theta^2 \genfrac{[}{]}{0pt}{}{10}{01} \theta^2 \genfrac{[}{]}{0pt}{}{01}{01} + \theta^2 \genfrac{[}{]}{0pt}{}{00}{10} \theta^2 \genfrac{[}{]}{0pt}{}{11}{11} & \left(\genfrac{[}{]}{0pt}{}{11}{01}\right) \\
\theta^2 \genfrac{[}{]}{0pt}{}{00}{00} \theta^2 \genfrac{[}{]}{0pt}{}{00}{11} = \theta^2 \genfrac{[}{]}{0pt}{}{11}{00} \theta^2 \genfrac{[}{]}{0pt}{}{11}{11} + \theta^2 \genfrac{[}{]}{0pt}{}{00}{10} \theta^2 \genfrac{[}{]}{0pt}{}{00}{01} & \left(\genfrac{[}{]}{0pt}{}{00}{11}\right) \\
\theta^2 \genfrac{[}{]}{0pt}{}{00}{11} \theta^2 \genfrac{[}{]}{0pt}{}{10}{00} = \theta^2 \genfrac{[}{]}{0pt}{}{01}{00} \theta^2 \genfrac{[}{]}{0pt}{}{11}{11} + \theta^2 \genfrac{[}{]}{0pt}{}{10}{01} \theta^2 \genfrac{[}{]}{0pt}{}{00}{10} & \left(\genfrac{[}{]}{0pt}{}{10}{11}\right) \\
\theta^2 \genfrac{[}{]}{0pt}{}{00}{11} \theta^2 \genfrac{[}{]}{0pt}{}{01}{00} = \theta^2 \genfrac{[}{]}{0pt}{}{10}{00} \theta^2 \genfrac{[}{]}{0pt}{}{11}{11} + \theta^2 \genfrac{[}{]}{0pt}{}{01}{10} \theta^2 \genfrac{[}{]}{0pt}{}{00}{01} & \left(\genfrac{[}{]}{0pt}{}{01}{11}\right) \\
\theta^2 \genfrac{[}{]}{0pt}{}{11}{00} \theta^2 \genfrac{[}{]}{0pt}{}{00}{11} = \theta^2 \genfrac{[}{]}{0pt}{}{00}{00} \theta^2 \genfrac{[}{]}{0pt}{}{11}{11} + \theta^2 \genfrac{[}{]}{0pt}{}{10}{01} \theta^2 \genfrac{[}{]}{0pt}{}{01}{10} & \left(\genfrac{[}{]}{0pt}{}{11}{11}\right).
\end{array}
$$

We denote $D([\varepsilon], [\delta]) = \theta_1[\varepsilon]\theta_2[\delta] - \theta_2[\varepsilon]\theta_1[\delta]$. Then the following Rosenhain formulae are valid:

$$
\begin{array}{ll}
D\left(\genfrac{[}{]}{0pt}{}{01}{01}, \genfrac{[}{]}{0pt}{}{11}{01}\right) = \pi^2\theta \genfrac{[}{]}{0pt}{}{00}{10} \theta \genfrac{[}{]}{0pt}{}{00}{11} \theta \genfrac{[}{]}{0pt}{}{11}{11} \theta \genfrac{[}{]}{0pt}{}{01}{10} & \left(\genfrac{[}{]}{0pt}{}{10}{00}\right) \\
D\left(\genfrac{[}{]}{0pt}{}{11}{10}, \genfrac{[}{]}{0pt}{}{10}{10}\right) = \pi^2\theta \genfrac{[}{]}{0pt}{}{11}{11} \theta \genfrac{[}{]}{0pt}{}{00}{01} \theta \genfrac{[}{]}{0pt}{}{00}{11} \theta \genfrac{[}{]}{0pt}{}{10}{01} & \left(\genfrac{[}{]}{0pt}{}{01}{00}\right) \\
D\left(\genfrac{[}{]}{0pt}{}{10}{11}, \genfrac{[}{]}{0pt}{}{01}{11}\right) = \pi^2\theta \genfrac{[}{]}{0pt}{}{01}{10} \theta \genfrac{[}{]}{0pt}{}{10}{01} \theta \genfrac{[}{]}{0pt}{}{00}{10} \theta \genfrac{[}{]}{0pt}{}{00}{01} & \left(\genfrac{[}{]}{0pt}{}{11}{00}\right) \\
D\left(\genfrac{[}{]}{0pt}{}{01}{11}, \genfrac{[}{]}{0pt}{}{01}{01}\right) = \pi^2\theta \genfrac{[}{]}{0pt}{}{10}{00} \theta \genfrac{[}{]}{0pt}{}{10}{01} \theta \genfrac{[}{]}{0pt}{}{11}{00} \theta \genfrac{[}{]}{0pt}{}{11}{11} & \left(\genfrac{[}{]}{0pt}{}{00}{10}\right) \\
D\left(\genfrac{[}{]}{0pt}{}{01}{11}, \genfrac{[}{]}{0pt}{}{11}{01}\right) = \pi^2\theta \genfrac{[}{]}{0pt}{}{00}{00} \theta \genfrac{[}{]}{0pt}{}{00}{01} \theta \genfrac{[}{]}{0pt}{}{11}{11} \theta \genfrac{[}{]}{0pt}{}{01}{00} & \left(\genfrac{[}{]}{0pt}{}{10}{10}\right) \\
D\left(\genfrac{[}{]}{0pt}{}{10}{11}, \genfrac{[}{]}{0pt}{}{11}{01}\right) = \pi^2\theta \genfrac{[}{]}{0pt}{}{11}{00} \theta \genfrac{[}{]}{0pt}{}{00}{11} \theta \genfrac{[}{]}{0pt}{}{00}{01} \theta \genfrac{[}{]}{0pt}{}{10}{00} & \left(\genfrac{[}{]}{0pt}{}{01}{10}\right) \\
D\left(\genfrac{[}{]}{0pt}{}{10}{11}, \genfrac{[}{]}{0pt}{}{01}{01}\right) = \pi^2\theta \genfrac{[}{]}{0pt}{}{01}{00} \theta \genfrac{[}{]}{0pt}{}{10}{01} \theta \genfrac{[}{]}{0pt}{}{00}{00} \theta \genfrac{[}{]}{0pt}{}{00}{11} & \left(\genfrac{[}{]}{0pt}{}{11}{10}\right) \\
D\left(\genfrac{[}{]}{0pt}{}{10}{10}, \genfrac{[}{]}{0pt}{}{10}{11}\right) = \pi^2\theta \genfrac{[}{]}{0pt}{}{01}{00} \theta \genfrac{[}{]}{0pt}{}{01}{10} \theta \genfrac{[}{]}{0pt}{}{11}{11} \theta \genfrac{[}{]}{0pt}{}{11}{00} & \left(\genfrac{[}{]}{0pt}{}{00}{01}\right) \\
D\left(\genfrac{[}{]}{0pt}{}{11}{10}, \genfrac{[}{]}{0pt}{}{01}{11}\right) = \pi^2\theta \genfrac{[}{]}{0pt}{}{00}{11} \theta \genfrac{[}{]}{0pt}{}{00}{10} \theta \genfrac{[}{]}{0pt}{}{11}{00} \theta \genfrac{[}{]}{0pt}{}{01}{00} & \left(\genfrac{[}{]}{0pt}{}{10}{01}\right) \\
D\left(\genfrac{[}{]}{0pt}{}{11}{10}, \genfrac{[}{]}{0pt}{}{10}{11}\right) = \pi^2\theta \genfrac{[}{]}{0pt}{}{11}{11} \theta \genfrac{[}{]}{0pt}{}{00}{00} \theta \genfrac{[}{]}{0pt}{}{00}{10} \theta \genfrac{[}{]}{0pt}{}{10}{00} & \left(\genfrac{[}{]}{0pt}{}{01}{01}\right) \\
D\left(\genfrac{[}{]}{0pt}{}{10}{10}, \genfrac{[}{]}{0pt}{}{01}{11}\right) = \pi^2\theta \genfrac{[}{]}{0pt}{}{01}{10} \theta \genfrac{[}{]}{0pt}{}{10}{00} \theta \genfrac{[}{]}{0pt}{}{00}{11} \theta \genfrac{[}{]}{0pt}{}{00}{00} & \left(\genfrac{[}{]}{0pt}{}{11}{01}\right) \\
D\left(\genfrac{[}{]}{0pt}{}{11}{10}, \genfrac{[}{]}{0pt}{}{11}{01}\right) = \pi^2\theta \genfrac{[}{]}{0pt}{}{10}{01} \theta \genfrac{[}{]}{0pt}{}{10}{00} \theta \genfrac{[}{]}{0pt}{}{01}{10} \theta \genfrac{[}{]}{0pt}{}{01}{00} & \left(\genfrac{[}{]}{0pt}{}{00}{11}\right) \\
D\left(\genfrac{[}{]}{0pt}{}{11}{10}, \genfrac{[}{]}{0pt}{}{01}{01}\right) = \pi^2\theta \genfrac{[}{]}{0pt}{}{00}{01} \theta \genfrac{[}{]}{0pt}{}{00}{00} \theta \genfrac{[}{]}{0pt}{}{11}{00} \theta \genfrac{[}{]}{0pt}{}{01}{10} & \left(\genfrac{[}{]}{0pt}{}{10}{11}\right) \\
D\left(\genfrac{[}{]}{0pt}{}{10}{10}, \genfrac{[}{]}{0pt}{}{11}{01}\right) = \pi^2\theta \genfrac{[}{]}{0pt}{}{11}{00} \theta \genfrac{[}{]}{0pt}{}{00}{10} \theta \genfrac{[}{]}{0pt}{}{00}{00} \theta \genfrac{[}{]}{0pt}{}{10}{01} & \left(\genfrac{[}{]}{0pt}{}{01}{11}\right) \\
D\left(\genfrac{[}{]}{0pt}{}{10}{10}, \genfrac{[}{]}{0pt}{}{01}{01}\right) = \pi^2\theta \genfrac{[}{]}{0pt}{}{01}{00} \theta \genfrac{[}{]}{0pt}{}{10}{00} \theta \genfrac{[}{]}{0pt}{}{00}{01} \theta \genfrac{[}{]}{0pt}{}{00}{10} & \left(\genfrac{[}{]}{0pt}{}{11}{11}\right).
\end{array}
$$

B.2. Addition theorem for second-order theta functions at $g = 2$

Here we give the expanded forms of (4.22). We introduce the notation $\hat{\theta}[\varepsilon](z) = \theta[\varepsilon](z|2\tau)$.

$$
\begin{aligned}
\theta\left[\begin{smallmatrix}00\\00\end{smallmatrix}\right]\theta\left[\begin{smallmatrix}00\\00\end{smallmatrix}\right](z) &= \hat\theta^2\left[\begin{smallmatrix}00\\00\end{smallmatrix}\right](z)+\hat\theta^2\left[\begin{smallmatrix}11\\00\end{smallmatrix}\right](z)+\hat\theta^2\left[\begin{smallmatrix}10\\00\end{smallmatrix}\right](z)+\hat\theta^2\left[\begin{smallmatrix}01\\00\end{smallmatrix}\right](z)\\
\theta\left[\begin{smallmatrix}00\\11\end{smallmatrix}\right]\theta\left[\begin{smallmatrix}00\\11\end{smallmatrix}\right](z) &= \hat\theta^2\left[\begin{smallmatrix}00\\00\end{smallmatrix}\right](z)+\hat\theta^2\left[\begin{smallmatrix}11\\00\end{smallmatrix}\right](z)-\hat\theta^2\left[\begin{smallmatrix}10\\00\end{smallmatrix}\right](z)-\hat\theta^2\left[\begin{smallmatrix}01\\00\end{smallmatrix}\right](z)\\
\theta\left[\begin{smallmatrix}00\\10\end{smallmatrix}\right]\theta\left[\begin{smallmatrix}00\\10\end{smallmatrix}\right](z) &= \hat\theta^2\left[\begin{smallmatrix}00\\00\end{smallmatrix}\right](z)-\hat\theta^2\left[\begin{smallmatrix}11\\00\end{smallmatrix}\right](z)-\hat\theta^2\left[\begin{smallmatrix}10\\00\end{smallmatrix}\right](z)+\hat\theta^2\left[\begin{smallmatrix}01\\00\end{smallmatrix}\right](z)\\
\theta\left[\begin{smallmatrix}00\\01\end{smallmatrix}\right]\theta\left[\begin{smallmatrix}00\\01\end{smallmatrix}\right](z) &= \hat\theta^2\left[\begin{smallmatrix}00\\00\end{smallmatrix}\right](z)-\hat\theta^2\left[\begin{smallmatrix}11\\00\end{smallmatrix}\right](z)+\hat\theta^2\left[\begin{smallmatrix}10\\00\end{smallmatrix}\right](z)-\hat\theta^2\left[\begin{smallmatrix}01\\00\end{smallmatrix}\right](z)
\end{aligned}
$$

$$
\begin{aligned}
\theta\left[\begin{smallmatrix}11\\00\end{smallmatrix}\right]\theta\left[\begin{smallmatrix}11\\00\end{smallmatrix}\right](z) &= 2\hat\theta\left[\begin{smallmatrix}11\\00\end{smallmatrix}\right](z)\hat\theta\left[\begin{smallmatrix}00\\00\end{smallmatrix}\right](z)+2\hat\theta\left[\begin{smallmatrix}10\\00\end{smallmatrix}\right](z)\hat\theta\left[\begin{smallmatrix}01\\00\end{smallmatrix}\right](z)\\
\theta\left[\begin{smallmatrix}11\\11\end{smallmatrix}\right]\theta\left[\begin{smallmatrix}11\\11\end{smallmatrix}\right](z) &= 2\hat\theta\left[\begin{smallmatrix}11\\00\end{smallmatrix}\right](z)\hat\theta\left[\begin{smallmatrix}00\\00\end{smallmatrix}\right](z)-2\hat\theta\left[\begin{smallmatrix}10\\00\end{smallmatrix}\right](z)\hat\theta\left[\begin{smallmatrix}01\\00\end{smallmatrix}\right](z)\\
\theta\left[\begin{smallmatrix}01\\00\end{smallmatrix}\right]\theta\left[\begin{smallmatrix}01\\00\end{smallmatrix}\right](z) &= 2\hat\theta\left[\begin{smallmatrix}01\\00\end{smallmatrix}\right](z)\hat\theta\left[\begin{smallmatrix}00\\00\end{smallmatrix}\right](z)+2\hat\theta\left[\begin{smallmatrix}10\\00\end{smallmatrix}\right](z)\hat\theta\left[\begin{smallmatrix}11\\00\end{smallmatrix}\right](z)\\
\theta\left[\begin{smallmatrix}01\\10\end{smallmatrix}\right]\theta\left[\begin{smallmatrix}01\\10\end{smallmatrix}\right](z) &= 2\hat\theta\left[\begin{smallmatrix}01\\00\end{smallmatrix}\right](z)\hat\theta\left[\begin{smallmatrix}00\\00\end{smallmatrix}\right](z)-2\hat\theta\left[\begin{smallmatrix}10\\00\end{smallmatrix}\right](z)\hat\theta\left[\begin{smallmatrix}11\\00\end{smallmatrix}\right](z)\\
\theta\left[\begin{smallmatrix}10\\00\end{smallmatrix}\right]\theta\left[\begin{smallmatrix}10\\00\end{smallmatrix}\right](z) &= 2\hat\theta\left[\begin{smallmatrix}10\\00\end{smallmatrix}\right](z)\hat\theta\left[\begin{smallmatrix}00\\00\end{smallmatrix}\right](z)+2\hat\theta\left[\begin{smallmatrix}01\\00\end{smallmatrix}\right](z)\hat\theta\left[\begin{smallmatrix}11\\00\end{smallmatrix}\right](z)\\
\theta\left[\begin{smallmatrix}10\\01\end{smallmatrix}\right]\theta\left[\begin{smallmatrix}10\\01\end{smallmatrix}\right](z) &= 2\hat\theta\left[\begin{smallmatrix}10\\00\end{smallmatrix}\right](z)\hat\theta\left[\begin{smallmatrix}00\\00\end{smallmatrix}\right](z)-2\hat\theta\left[\begin{smallmatrix}01\\00\end{smallmatrix}\right](z)\hat\theta\left[\begin{smallmatrix}11\\00\end{smallmatrix}\right](z)
\end{aligned}
$$

$$
\begin{aligned}
\theta\left[\begin{smallmatrix}11\\00\end{smallmatrix}\right]\theta\left[\begin{smallmatrix}11\\01\end{smallmatrix}\right](z) &= 2\hat\theta\left[\begin{smallmatrix}10\\01\end{smallmatrix}\right](z)\hat\theta\left[\begin{smallmatrix}01\\01\end{smallmatrix}\right](z)+2\hat\theta\left[\begin{smallmatrix}11\\01\end{smallmatrix}\right](z)\hat\theta\left[\begin{smallmatrix}00\\01\end{smallmatrix}\right](z)\\
\theta\left[\begin{smallmatrix}11\\11\end{smallmatrix}\right]\theta\left[\begin{smallmatrix}11\\10\end{smallmatrix}\right](z) &= 2\hat\theta\left[\begin{smallmatrix}10\\01\end{smallmatrix}\right](z)\hat\theta\left[\begin{smallmatrix}01\\01\end{smallmatrix}\right](z)-2\hat\theta\left[\begin{smallmatrix}11\\01\end{smallmatrix}\right](z)\hat\theta\left[\begin{smallmatrix}00\\01\end{smallmatrix}\right](z)\\
\theta\left[\begin{smallmatrix}01\\00\end{smallmatrix}\right]\theta\left[\begin{smallmatrix}01\\01\end{smallmatrix}\right](z) &= 2\hat\theta\left[\begin{smallmatrix}01\\01\end{smallmatrix}\right](z)\hat\theta\left[\begin{smallmatrix}00\\01\end{smallmatrix}\right](z)+2\hat\theta\left[\begin{smallmatrix}10\\01\end{smallmatrix}\right](z)\hat\theta\left[\begin{smallmatrix}11\\01\end{smallmatrix}\right](z)\\
\theta\left[\begin{smallmatrix}01\\10\end{smallmatrix}\right]\theta\left[\begin{smallmatrix}01\\11\end{smallmatrix}\right](z) &= 2\hat\theta\left[\begin{smallmatrix}01\\01\end{smallmatrix}\right](z)\hat\theta\left[\begin{smallmatrix}00\\01\end{smallmatrix}\right](z)-2\hat\theta\left[\begin{smallmatrix}10\\01\end{smallmatrix}\right](z)\hat\theta\left[\begin{smallmatrix}11\\01\end{smallmatrix}\right](z)\\
\theta\left[\begin{smallmatrix}10\\00\end{smallmatrix}\right]\theta\left[\begin{smallmatrix}10\\10\end{smallmatrix}\right](z) &= 2\hat\theta\left[\begin{smallmatrix}00\\10\end{smallmatrix}\right](z)\hat\theta\left[\begin{smallmatrix}10\\10\end{smallmatrix}\right](z)+2\hat\theta\left[\begin{smallmatrix}11\\10\end{smallmatrix}\right](z)\hat\theta\left[\begin{smallmatrix}01\\10\end{smallmatrix}\right](z)\\
\theta\left[\begin{smallmatrix}10\\01\end{smallmatrix}\right]\theta\left[\begin{smallmatrix}10\\11\end{smallmatrix}\right](z) &= 2\hat\theta\left[\begin{smallmatrix}00\\10\end{smallmatrix}\right](z)\hat\theta\left[\begin{smallmatrix}10\\10\end{smallmatrix}\right](z)-2\hat\theta\left[\begin{smallmatrix}11\\10\end{smallmatrix}\right](z)\hat\theta\left[\begin{smallmatrix}01\\10\end{smallmatrix}\right](z)
\end{aligned}
$$

$$
\begin{aligned}
\theta\left[\begin{smallmatrix}11\\00\end{smallmatrix}\right]\theta_k\left[\begin{smallmatrix}11\\01\end{smallmatrix}\right] &= 2\hat\theta\left[\begin{smallmatrix}00\\01\end{smallmatrix}\right]\hat\theta_k\left[\begin{smallmatrix}11\\01\end{smallmatrix}\right]+2\hat\theta\left[\begin{smallmatrix}10\\01\end{smallmatrix}\right]\hat\theta_k\left[\begin{smallmatrix}01\\01\end{smallmatrix}\right]\\
\theta\left[\begin{smallmatrix}11\\11\end{smallmatrix}\right]\theta_k\left[\begin{smallmatrix}11\\10\end{smallmatrix}\right] &= 2\hat\theta\left[\begin{smallmatrix}10\\01\end{smallmatrix}\right]\hat\theta_k\left[\begin{smallmatrix}01\\01\end{smallmatrix}\right]-2\hat\theta\left[\begin{smallmatrix}00\\01\end{smallmatrix}\right]\hat\theta_k\left[\begin{smallmatrix}11\\01\end{smallmatrix}\right]\\
\theta\left[\begin{smallmatrix}01\\00\end{smallmatrix}\right]\theta_k\left[\begin{smallmatrix}01\\01\end{smallmatrix}\right] &= 2\hat\theta\left[\begin{smallmatrix}00\\01\end{smallmatrix}\right]\hat\theta_k\left[\begin{smallmatrix}01\\01\end{smallmatrix}\right]+2\hat\theta\left[\begin{smallmatrix}10\\01\end{smallmatrix}\right]\hat\theta_k\left[\begin{smallmatrix}11\\01\end{smallmatrix}\right]\\
\theta\left[\begin{smallmatrix}01\\10\end{smallmatrix}\right]\theta_k\left[\begin{smallmatrix}01\\11\end{smallmatrix}\right] &= 2\hat\theta\left[\begin{smallmatrix}00\\01\end{smallmatrix}\right]\hat\theta_k\left[\begin{smallmatrix}01\\01\end{smallmatrix}\right]-2\hat\theta\left[\begin{smallmatrix}10\\01\end{smallmatrix}\right]\hat\theta_k\left[\begin{smallmatrix}11\\01\end{smallmatrix}\right]\\
\theta\left[\begin{smallmatrix}10\\00\end{smallmatrix}\right]\theta_k\left[\begin{smallmatrix}10\\10\end{smallmatrix}\right] &= 2\hat\theta\left[\begin{smallmatrix}00\\10\end{smallmatrix}\right]\hat\theta_k\left[\begin{smallmatrix}10\\10\end{smallmatrix}\right]+2\hat\theta\left[\begin{smallmatrix}01\\10\end{smallmatrix}\right]\hat\theta_k\left[\begin{smallmatrix}11\\10\end{smallmatrix}\right]\\
\theta\left[\begin{smallmatrix}10\\01\end{smallmatrix}\right]\theta_k\left[\begin{smallmatrix}10\\11\end{smallmatrix}\right] &= 2\hat\theta\left[\begin{smallmatrix}00\\10\end{smallmatrix}\right]\hat\theta_k\left[\begin{smallmatrix}10\\10\end{smallmatrix}\right]-2\hat\theta\left[\begin{smallmatrix}01\\10\end{smallmatrix}\right]\hat\theta_k\left[\begin{smallmatrix}11\\10\end{smallmatrix}\right].
\end{aligned}
$$

B.3. Theta constants of 2^p-sheeted coverings over a torus

In this subsection we denote the Jacobi theta constants by $\vartheta_j = \vartheta_j(0|2^p\tau_{11})$, $\widetilde{\vartheta}_j = \vartheta_j(0|2^p\tau_{22})$, $j = 2, 3, 4$.

$p = 1$. Let $\tau = \begin{pmatrix}\tau_{11} & \frac12\\ \frac12 & \tau_{22}\end{pmatrix}$. Then

$$
\begin{aligned}
&\theta\left[\begin{smallmatrix}10\\00\end{smallmatrix}\right] = \theta\left[\begin{smallmatrix}10\\01\end{smallmatrix}\right] = (2\vartheta_2\vartheta_3\widetilde{\vartheta}_3\widetilde{\vartheta}_4)^{1/2} && \theta\left[\begin{smallmatrix}01\\10\end{smallmatrix}\right] = \theta\left[\begin{smallmatrix}01\\00\end{smallmatrix}\right] = (2\vartheta_3\vartheta_4\widetilde{\vartheta}_2\widetilde{\vartheta}_3)^{1/2}\\
&\theta\left[\begin{smallmatrix}11\\00\end{smallmatrix}\right] = -\mathrm{i}\theta\left[\begin{smallmatrix}11\\11\end{smallmatrix}\right] = (2\vartheta_2\vartheta_4\widetilde{\vartheta}_2\widetilde{\vartheta}_4)^{1/2} &&\\
&\theta\left[\begin{smallmatrix}00\\00\end{smallmatrix}\right] = (\vartheta_3^2\widetilde{\vartheta}_3^2+\vartheta_2^2\widetilde{\vartheta}_4^2+\vartheta_4^2\widetilde{\vartheta}_2^2)^{1/2} && \theta\left[\begin{smallmatrix}00\\11\end{smallmatrix}\right] = (\vartheta_3^2\widetilde{\vartheta}_3^2-\vartheta_2^2\widetilde{\vartheta}_4^2-\vartheta_4^2\widetilde{\vartheta}_2^2)^{1/2}\\
&\theta\left[\begin{smallmatrix}00\\10\end{smallmatrix}\right] = (\vartheta_3^2\widetilde{\vartheta}_3^2-\vartheta_2^2\widetilde{\vartheta}_4^2+\vartheta_4^2\widetilde{\vartheta}_2^2)^{1/2} && \theta\left[\begin{smallmatrix}00\\01\end{smallmatrix}\right] = (\vartheta_3^2\widetilde{\vartheta}_3^2+\vartheta_2^2\widetilde{\vartheta}_4^2-\vartheta_4^2\widetilde{\vartheta}_2^2)^{1/2}\\
&\theta_1\left[\begin{smallmatrix}11\\10\end{smallmatrix}\right] = -\pi\theta\left[\begin{smallmatrix}11\\00\end{smallmatrix}\right]\vartheta_3^2 && \theta_2\left[\begin{smallmatrix}11\\10\end{smallmatrix}\right] = -\mathrm{i}\pi\theta\left[\begin{smallmatrix}11\\11\end{smallmatrix}\right]\widetilde{\vartheta}_3^2\\
&\theta_1\left[\begin{smallmatrix}11\\01\end{smallmatrix}\right] = -\mathrm{i}\pi\theta\left[\begin{smallmatrix}11\\00\end{smallmatrix}\right]\vartheta_3^2 && \theta_2\left[\begin{smallmatrix}11\\01\end{smallmatrix}\right] = -\pi\theta\left[\begin{smallmatrix}11\\00\end{smallmatrix}\right]\widetilde{\vartheta}_3^2\\
&\theta_1\left[\begin{smallmatrix}01\\01\end{smallmatrix}\right] = -\mathrm{i}\pi\theta\left[\begin{smallmatrix}01\\00\end{smallmatrix}\right]\vartheta_2^2 && \theta_2\left[\begin{smallmatrix}01\\01\end{smallmatrix}\right] = -\pi\theta\left[\begin{smallmatrix}01\\01\end{smallmatrix}\right]\widetilde{\vartheta}_4^2\\
&\theta_1\left[\begin{smallmatrix}01\\11\end{smallmatrix}\right] = \mathrm{i}\pi\theta\left[\begin{smallmatrix}01\\10\end{smallmatrix}\right]\vartheta_2^2 && \theta_2\left[\begin{smallmatrix}01\\11\end{smallmatrix}\right] = -\pi\theta\left[\begin{smallmatrix}01\\10\end{smallmatrix}\right]\widetilde{\vartheta}_4^2
\end{aligned}
$$

$$\theta_1\begin{bmatrix}10\\11\end{bmatrix} = -\pi\theta\begin{bmatrix}10\\01\end{bmatrix}\vartheta_4^2 \qquad \theta_2\begin{bmatrix}10\\11\end{bmatrix} = \mathrm{i}\pi\theta\begin{bmatrix}10\\01\end{bmatrix}\widetilde{\vartheta}_2^2$$
$$\theta_1\begin{bmatrix}10\\10\end{bmatrix} = -\pi\theta\begin{bmatrix}10\\00\end{bmatrix}\vartheta_4^2 \qquad \theta_2\begin{bmatrix}10\\10\end{bmatrix} = -\mathrm{i}\pi\theta\begin{bmatrix}10\\00\end{bmatrix}\widetilde{\vartheta}_2^2 .$$

$p=2$. Let $\tau = \begin{pmatrix}\tau_{11} & \frac14\\ \frac14 & \tau_{22}\end{pmatrix}$ and denote $X=\vartheta_3\widetilde{\vartheta}_3$, $Y=\vartheta_2\widetilde{\vartheta}_4$, $Z=\vartheta_4\widetilde{\vartheta}_2$, $A=-X^2+Y^2+Z^2$, $B=X^2-Y^2+Z^2$, $C=X^2+Y^2-Z^2$, $D=A+B+C$. Then the following formulae hold:

$$\theta\begin{bmatrix}00\\00\end{bmatrix} = X+Y+Z \qquad \theta\begin{bmatrix}00\\01\end{bmatrix} = X+Y-Z$$
$$\theta\begin{bmatrix}00\\10\end{bmatrix} = X-Y+Z \qquad \theta\begin{bmatrix}00\\11\end{bmatrix} = X-Y-Z$$
$$\theta^2\begin{bmatrix}10\\00\end{bmatrix} = 2^{3/2}(XY)^{1/2}(D^{1/2}+2^{1/2}Z) \qquad \theta^2\begin{bmatrix}10\\01\end{bmatrix} = 2^{3/2}(XY)^{1/2}(D^{1/2}-2^{1/2}Z)$$
$$\theta^2\begin{bmatrix}01\\00\end{bmatrix} = 2^{3/2}(XZ)^{1/2}(D^{1/2}+2^{1/2}Y) \qquad \theta^2\begin{bmatrix}01\\10\end{bmatrix} = 2^{3/2}(XZ)^{1/2}(D^{1/2}-2^{1/2}Y)$$
$$\theta^2\begin{bmatrix}11\\00\end{bmatrix} = 2^{3/2}(YZ)^{1/2}(D^{1/2}+2^{1/2}X) \qquad \theta^2\begin{bmatrix}11\\11\end{bmatrix} = 2^{3/2}(YZ)^{1/2}(D^{1/2}-2^{1/2}X) .$$

$$\theta_1\begin{bmatrix}10\\10\end{bmatrix} = -\pi(2XY)^{1/4}(\vartheta_4^2B^{1/2}+2^{1/2}\vartheta_3^2Z)(D^{1/2}+2^{1/2}Z)^{-1/2}$$
$$\theta_2\begin{bmatrix}10\\10\end{bmatrix} = -\mathrm{i}\pi(2XY)^{1/4}(\widetilde{\vartheta}_2^2B^{1/2}+2^{1/2}\widetilde{\vartheta}_3^2Z)(D^{1/2}+2^{1/2}Z)^{-1/2}$$
$$\theta_1\begin{bmatrix}01\\01\end{bmatrix} = -\mathrm{i}\pi(2XZ)^{1/4}(\vartheta_2^2C^{1/2}+2^{1/2}\vartheta_3^2Y)(D^{1/2}+2^{1/2}Y)^{-1/2}$$
$$\theta_2\begin{bmatrix}01\\01\end{bmatrix} = -\pi(2XZ)^{1/4}(\widetilde{\vartheta}_4^2C^{1/2}+2^{1/2}\widetilde{\vartheta}_3^2Y)(D^{1/2}+2^{1/2}Y)^{-1/2}$$
$$\theta_1\begin{bmatrix}11\\01\end{bmatrix} = -\mathrm{i}\pi(2ZY)^{1/4}(\vartheta_3^2C^{1/2}+2^{1/2}\vartheta_2^2X)(D^{1/2}+2^{1/2}X)^{-1/2}$$
$$\theta_2\begin{bmatrix}11\\01\end{bmatrix} = -\pi(2ZY)^{1/4}(\widetilde{\vartheta}_3^2C^{1/2}+2^{1/2}\widetilde{\vartheta}_4^2X)(D^{1/2}+2^{1/2}X)^{-1/2}$$
$$\theta_1\begin{bmatrix}11\\10\end{bmatrix} = -\pi(2ZY)^{1/4}(\vartheta_3^2B^{1/2}+2^{1/2}\vartheta_4^2X)(D^{1/2}+2^{1/2}X)^{-1/2}$$
$$\theta_2\begin{bmatrix}11\\10\end{bmatrix} = -\pi(2ZY)^{1/4}(\widetilde{\vartheta}_3^2B^{1/2}+2^{1/2}\widetilde{\vartheta}_4^2X)(D^{1/2}+2^{1/2}X)^{-1/2}$$
$$\theta_1\begin{bmatrix}10\\11\end{bmatrix} = -\mathrm{i}\pi(2XY)^{1/4}(\vartheta_4^2A^{1/2}-2^{1/2}\mathrm{i}\vartheta_2^2Z)(D^{1/2}+2^{1/2}Z)^{-1/2}$$
$$\theta_2\begin{bmatrix}10\\11\end{bmatrix} = -\mathrm{i}\pi(2XY)^{1/4}(\widetilde{\vartheta}_2^2A^{1/2}-2^{1/2}\mathrm{i}\widetilde{\vartheta}_3^2Z)(D^{1/2}+2^{1/2}Z)^{-1/2}$$
$$\theta_1\begin{bmatrix}01\\11\end{bmatrix} = \pi(2XZ)^{1/4}(\vartheta_2^2A^{1/2}-2^{1/2}\mathrm{i}\vartheta_3^2Y)(D^{1/2}+2^{1/2}Y)^{-1/2}$$
$$\theta_2\begin{bmatrix}01\\11\end{bmatrix} = -\mathrm{i}\pi(2XZ)^{1/4}(\widetilde{\vartheta}_4^2A^{1/2}-2^{1/2}\mathrm{i}\widetilde{\vartheta}_3^2Y)(D^{1/2}+2^{1/2}Y)^{-1/2} .$$

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